Defaulting firms and systemic risks in financial networks: A normative approach

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Abstract

We study systemic risk in an interbank market, employing an explicit axiomatization inspired by Eisenberg and Noe (2001) and Rogers and Veraart (2013). Instead of focusing on a clearing payment scheme, we characterize the smallest (in the sense of inclusion) set of ex-post defaulting firms. This novel approach allows us to analyze the normative implications of the Eisenberg-Noe axioms. We first show that the Absolute Priority axiom, which states that defaulting firms must end up with zero net worth, has no impact on minimal default sets. Second, relaxing the Limited Payments axiom, which can be interpreted as allowing a central planner to transfer resources from rich firms to poor, does not further reduce the minimal default sets, although other default sets are possible. Our normative analysis sheds new light on the possible impacts of clearing mechanisms on default outcomes.

Key-Words: Credit risk, Systemic risk, Clearing system, Financial system.

JEL Classification: G21, G32, G33.

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1 Introduction

The 2007-2008 crisis and its aftermath have underlined the importance of the interconnections between financial institutions. On the one hand, regulators clearly had an imperfect understanding of the liabilities that institutions owed one another. A prominent example of this limited knowledge is the credit default swap market (which was a non-regulated, over-the-counter market) and the subsequent rescue of the insurance company AIG (for an exhaustive analysis of the roots of the 2007–2008 crisis, see the Financial Crisis Inquiry Commission 2011 report). On the other hand, little was – and probably still partly is – known about the impact of financial interconnections on financial stability and on the magnitude of so-called *systemic risk*. Interconnections have double-edged consequences. Standard portfolio arguments imply that interconnections may favor risk sharing, thereby generating a positive impact. Conversely, every interconnection can be seen as a new channel likely to favor the transmission of shocks, therefore feeding systemic risk.[1]

This paper studies the latter aspect of interconnections and focuses on an atemporal network model à la Eisenberg and Noe (2001). Interbank obligations are given and the objective is to determine a possible collection of inter-institutional payments – the so-called *Clearing Payment Matrix* – that fulfills a given set of constraints, which are supposed to "satisfy the standard conditions imposed by bankruptcy law" (Eisenberg and Noe, 2001). A huge literature has been built on the work of Eisenberg and Noe (2001) and has addressed a number of questions. For instance, how can we measure the vulnerability of a given network (Glasserman and Young, 2015; Demange, 2018)? To what extent do asset prices affect contagion in the event of a fire sale (Cifuentes et al., 2005)? How does network topology shape systemic risk (Acemoglu et al., 2015)? How should we account for multiple

[1]Recent surveys of contagion in financial markets include Allen and Babus (2009), Summer (2013), Cabrales et al. (2016), Hüser (2015), and Glasserman and Young (2016), among others.

debtors and creditors (Stutzer, 2018)? Do network dynamics affect contagion (Capponi and Chen, 2015 and Banerjee et al., 2018)?

In this paper, we examine the consequences of the axiomatic approach proposed by Eisenberg and Noe (2001) on a set of defaulting banks (rather than on the clearing payment matrix (CPM, henceforth), which is the approach generally taken in the literature). In particular, we focus on the smallest (in the sense of inclusion) set of defaulting banks that can be achieved given initial inter-bank liabilities. This set obviously characterizes the smallest number of defaulting firms, which is the criterion used by Nier et al. (2007) and Acemoglu et al. (2015), for instance.[2] However, the minimal set of defaulting banks provides much richer information than its cardinal, since it allows us to identify the defaulting banks and therefore their size and the extent of their interconnections. The identity of defaulting banks is of primary importance for normative purposes, such as regulatory and policy decisions, including bail-outs. Recall that in the 2007-2008 crisis, AIG was bailed out to avoid default contagion spreading to its counterparties and to reduce the risk of a systemic event. A couple of weeks before, in September 2008, Lehman Brothers, which was the fourth largest investment bank in the US at the time, had filed for bankruptcy. Whether Lehman Brothers should have been bailed out or not remains a puzzle. However, individual institutions' identities and their respective positions in the financial sector were key factors in such bail-out decisions. On a related topic, our notion of minimal defaulting firms also echoes the work of the Financial Stability Board (FSB) regarding *global systemically important financial institutions* (G-SIFIs), i.e. financial institutions whose failure is likely to put the whole financial system at risk, thereby endangering the global economy. One of the difficulties of the FSB's task is identifying these systemic financial institutions. In 2011, the FSB and the Basel Committee on Banking Supervision, with the help of na-

[2] Note that since the total number of banks is exogenous, this number is equivalent to the minimal share of defaulting banks.

tional authorities, determined a list of *global systemically important banks* (G-SIBs). An updated list is released every November (see Financial Stability Board, 2018a) and the identification methodology is public (see Basel Committee on Banking Supervision, 2013). A similar list of *global systemically important insurers* (G-SII) was published 2013 and is established with the help of the International Association of Insurance Supervisors. However, due to developments in the evaluation methodology, no list has been published since 2017, with no new list expected before 2022 (see Financial Stability Board, 2018b). How does this concept of systemic risk relate to our notion of minimal defaulting firms? First, although the sets of minimal defaulting firms and of G-SIFIs (or G-SIBs) can obviously differ, the concept of minimal defaulting firms can be useful for identifying systemic institutions and can complete other tools by providing an additional and complementary perspective on the systemic risk dimension. Second, these systemic institutions are subject to specific macro-prudential rules that notably include "more intensive and effective supervision" (Financial Stability Board, 2011) and they are grouped into several "buckets" in function of the degree of systemic risk. Minimal defaulting firms could perhaps constitute a specific bucket that also deserves special treatment. Overall, working with the minimal default set rather than the CPM allows us to adopt a normative approach rather than a positive one, which implies possible connections with macro-prudential supervision.

Our analysis is based on four axioms: *Limited Liability*, *Absolute Priority*, *Proportionality*, and *Limited Payments.* The first three axioms are explicitly stated in Eisenberg and Noe (2001), while the last one is implicit. As explained above, our first contribution is to use the concept of a minimal default set to investigate the consequences of these four axioms. More precisely, we study whether removing one or several axioms yields a smaller minimal default set. Our second contribution is to show that the Absolute Priority axiom (where defaulting firms are forced to pay all their assets to their creditors and end up with zero worth) is not independent of the other three axioms, in the sense that any minimal default set for a CPM satisfying the four axioms is also a minimal default set when Absolute Priority is relaxed – and the other way around. This result extends to non-zero default costs, as in Rogers and Veraart (2013). From a normative point of view, the Absolute Priority axiom can be removed with no consequence.

Last, we show that removing both the Absolute Priority axiom and the Limited Payments axiom (which prevents firms paying more than their initial liabilities) has a more subtle consequence. On the one hand, removing the Limited Payments condition does not further reduce the set of defaulting firms when there is no default cost, as in the initial Eisenberg and Noe (2001) framework. On the other hand, minimal default sets exist that satisfy Limited Liability and Proportionality but not Limited Payments. Relaxing the latter axiom can lead to minimal default sets that differ from those of the initial framework. Removing the Limited Payments axiom enables a "central planner" to transfer resources from rich firms to poor and can therefore be interpreted as moving from a decentralized market clearing system towards the centralized resolution of defaults. From a normative viewpoint, relaxing the Limited Payments axiom does not help shrink the set of defaulting firms. The set of defaulting firms can only be modified at the expense of triggering the default of firms that would not have defaulted in the original Eisenberg-Noe set-up.

The remainder of this paper is structured as follows. We present our model and explicit axioms in Section 2. We provide our results in Section 3. Section 4 concludes.

2 Model

2.1 Set-up

The set-up we consider builds on that of Eisenberg and Noe (2001). The main difference is that we allow for default costs, in a similar vein to Rogers and Veraart (2013). Within this framework, the financial entities and their obligations to one another are given.

We consider a set $\mathcal{N}$ of N financial entities, which we will simply refer to as firms, indexed by $i \in \mathcal{N}$. Each firm i is initially endowed with a nonnegative operating cash flow $e_i \geq 0$. We denote by $e = (e_i)_{i\in\mathcal{N}}$ the vector of operating cash flows that is constrained to belong to the nonnegative orthant $\mathcal{E} = \mathbb{R}_+^N$. Note that the quantity e_i could alternatively be interpreted as the net outside position of firm i after a firm-specific shock. This last interpretation is a variant of the Eisenberg-Noe framework proposed in Elsinger (2009) and used in Glasserman and Young (2015), for instance.

Firms are interconnected and the initial nominal liability matrix is denoted by $L = (L_{ij})_{i,j\in\mathcal{N}} \in \mathbb{R}_+^{N\times N}$. The quantity L_{ij} represents the nominal payment that firm i has to make to firm j. By construction, $L_{ii} = 0$ for any firm i. The set of all possible liability matrices is denoted by $\mathcal{L}$.

For the rest of the paper, we adopt the following notation. The sum of all payments that firm i has to make is denoted by $L_{i,\mathcal{N}} = \sum_{j\in\mathcal{N}} L_{ij}$. From an accounting perspective, this represents the (interbank) liabilities of firm i. Symmetrically, the sum of all payments due to firm i is denoted by $L_{\mathcal{N},i} = \sum_{j\in\mathcal{N}} L_{ji}$. This quantity represents the (interbank) assets of firm i. We then deduce that the ex-ante net worth of firm i equals $e_i + L_{\mathcal{N},i} - L_{i,\mathcal{N}}$. The net worth is said to be ex-ante since it depends on the initial liability matrix, but not on the actual payments made.

A CPM, denoted by $X \in \mathcal{L}$, gathers the actual nominal payments made between the different financial entities. The quantity X_{ij} is the clearing payment made by firm i to firm j. We define the quantities $X_{\mathcal{N},i}$ and $X_{i,\mathcal{N}}$ similarly to $L_{\mathcal{N},i}$ and $L_{i,\mathcal{N}}$. A firm i receives the total payment $X_{\mathcal{N},i}$ from all other firms, while it pays a global amount $X_{i,\mathcal{N}}$ to the other firms. An acceptable CPM should verify a set of rules, as formalized below in our axioms. Designing a CPM is not an easy task since the payments made by firm i depend on the payments made by other firms to firm i, which in turn depend on the payments of i. Eisenberg and Noe (2001) use a fixed-point argument to show the existence of a CPM. We consider that a firm i defaults if one of its clearing payments differs from its corresponding liability. Formally, firm i defaults if $X_{ij} \neq L_{ij}$ for some firm j. For a given liability matrix L and a given CPM X, the set of defaulting firms will be denoted by $\mathcal{D}(L, X)$ and formally defined as:

$$\mathcal{D}(L, X) = \{i \in \mathcal{N}, \exists j \in \mathcal{N}, X_{ij} \neq L_{ij}\}.$$

Note that we qualify a firm as defaulting when payments differ from actual liabilities. We introduce below the explicit axiom that total clearing payments must be smaller than initial liabilities.[3] Interestingly, as we explain in Section 3.4, the actual default mechanisms employed by clearing houses may feature clearing payments higher than initial liabilities. We call $\overline{\mathcal{D}}(L, X)$ the set of firms that do not default, *i.e.*,

$$\overline{\mathcal{D}}(L, X) = \mathcal{N} \setminus \mathcal{D}(L, X). \tag{1}$$

Finally, we consider that a firm is level-0-defaulting if the value of its initial net worth is negative. In other words, a firm will be said to be level-0-defaulting if its total liabilities exceed the total value of its assets (including its initial endowment). For a given vector e of initial endowments and a given liability matrix L, we denote by $\mathcal{D}_0(e, L)$ the set of level-0-defaulting firms that is formally defined as:

$$\mathcal{D}_0(e, L) = \{i \in \mathcal{N}, L_{\mathcal{N},i} + e_i - L_{i,\mathcal{N}} < 0\}.$$

2.2 Axioms

We now present the set of conditions that we impose on any acceptable CPM. Most of these conditions were introduced, at least implicitly, in Eisenberg and Noe (2001). Additionally, we assume that default is not costless and we introduce a two-dimensional explicit default cost, as in Rogers and Veraart (2013). First, the firm i is able to recover only a fraction $\alpha \in [0, 1]$ of the initial endowment e_i. The complement, which is not recovered,

[3]This type of default criterion is common in the bankruptcy literature, see Araujo and Páscoa (2002), Modica et al. (1998), and Eichberger et al. (2014), for instance.

is the first component of the default cost. Second, the actual liquidation value of firm i's asset can be smaller than its face value. The firm receives only a fraction $\beta \in [0,1]$ of the face value of its assets. The loss amounting to the share $1-\beta$ of the face value is the second component of the default cost, corresponding, for instance, to a fire sale (see Cifuentes et al. 2005 for explicit modeling of a fire sale in a contagion model).

We now introduce the four main axioms considered. In the remainder of the section, we consider as given an endowment vector $e = (e_j)_{j\in\mathcal{N}} \in \mathcal{E}$, a liability matrix $L = (L_{ij})_{i,j\in\mathcal{N}} \in \mathcal{L}$, and a CPM $X = (X_{ij})_{i,j\in\mathcal{N}} \in \mathcal{L}$.

Axiom 1 (Limited Liability, LL)

The CPM X satisfies Limited Liability if:

$$\begin{cases} \forall i \in \overline{\mathcal{D}}(L,X), X_{i,\mathcal{N}} \leq X_{\mathcal{N},i} + e_i \text{ and,} \\ \forall i \in \mathcal{D}(L,X), X_{i,\mathcal{N}} \leq \beta\, X_{\mathcal{N},i} + \alpha\, e_i. \end{cases}$$

In other words, Limited Liability prevents a firm paying more than it receives, regardless of whether the firm defaults or not. Recall that the quantity $X_{i,\mathcal{N}}$ represents the sum of all payments made by firm i, while $X_{\mathcal{N},i}$ represents the total payments made to firm i, and e_i its initial endowments. For a non-defaulting firm i, its resources amount to $X_{\mathcal{N},i} + e_i$ and its ex-post net worth, corresponding to CPM X, is therefore equal to $X_{\mathcal{N},i} + e_i - X_{i,\mathcal{N}}$. The net worth is said to be ex-post, to underline its dependence on the implementation of the CPM X and to distinguish it from the ex-ante net worth $e_i + L_{\mathcal{N},i} - L_{i,\mathcal{N}}$, which depends solely on the initial liability matrix, regardless of actual payments. For a defaulting firm i, default costs affect endowments and payments received, such that total resources amount to $\beta\, X_{\mathcal{N},i} + \alpha\, e_i$. The ex-post net worth of a defaulting firm i, which corresponds to CPM X, is therefore equal to $\beta\, X_{\mathcal{N},i} + \alpha\, e_i - X_{i,\mathcal{N}}$. Limited Liability then implies that the clearing payments system cannot lead to a situation where a firm has ex-post negative net worth, regardless of whether it is defaulting or not.

Axiom 2 (Absolute Priority, AP)

The CPM X satisfies Absolute Priority if:

$$\forall i \in \mathcal{D}(L, X), X_{i,\mathcal{N}} = \beta\, X_{\mathcal{N},i} + \alpha\, e_i.$$

Absolute Priority reinforces Limited Liability and requires that defaulting firms end up with exactly zero ex-post net worth, as defined above. This axiom guarantees that a defaulting firm does not keep resources that could be used to compensate its creditors, whose obligations are, by construction, not fully met.

Axiom 3 (Proportionality, P)

The CPM X satisfies Proportionality if:

$$\forall i, j \in \mathcal{N}, \begin{cases} L_{i,\mathcal{N}} > 0 \Rightarrow X_{ij} = \dfrac{L_{ij}}{L_{i,\mathcal{N}}} X_{i,\mathcal{N}} \text{ and,} \\ L_{i,\mathcal{N}} = 0 \Rightarrow X_{ij} = 0. \end{cases}$$

Proportionality designs the shapes of clearing payments, which are constrained to be proportional to actual liabilities. In the absence of default, the firm pays out its exact obligations and the axiom holds. Conversely, for a defaulting firm, the axiom implies that all claims have the same seniority and that all creditors should be equally served. See Elsinger (2009) or Gourieroux et al. (2013) for the introduction of different seniorities.

Axiom 4 (Limited Payments, LP)

The CPM X satisfies Limited Payments if:

$$\forall i \in \mathcal{N}, X_{i,\mathcal{N}} \leq L_{i,\mathcal{N}}.$$

Limited Payments is our last axiom and it specifies that a firm's total clearing payment must not exceed its total nominal liability. This axiom is noticeably weaker than requiring every individual clearing payment to be smaller than every liability payment. However, combining Proportionality with Limited Payments implies that $X_{ij} \leq L_{ij}$. Using the definition of defaulting firms in equation (1), we deduce that a firm defaults if one of its clearing payments is smaller than the corresponding liability, i.e. if $X_{ij} < L_{ij}$ for some firm j. Our definition of default coupled with Limited Payments and Proportionality is the same as the "standard" definition of default that can be found in Eisenberg and Noe (2001) and Rogers and Veraart (2013), among many others.

We now turn to the definition of a solution in our set-up. Since we consider different combinations of axioms in our results, a solution will be defined subject to a set of axioms denoted Γ. An example of such a set is $\{LL, AP, P, LP\}$, which corresponds to the four axioms stated above.

Definition 1 (Solution set)

Let Γ be a set of axioms. The set $\mathcal{D} \subseteq \mathcal{N}$ is a Γ-solution to (e, L) if a CPM $X \in \mathcal{L}$ exists such that:

1. *X satisfies axioms in Γ, and*
2. *$\mathcal{D} = \mathcal{D}(L, X)$.*

The Γ-solution set $\mathcal{D}$ will be said to be minimal if the following implication holds:

$$\mathcal{D}' \text{ is a } \Gamma\text{-solution to } (e, L) \text{ and } \mathcal{D}' \subseteq \mathcal{D} \;\Rightarrow\; \mathcal{D} = \mathcal{D}'.$$

When there is no ambiguity on the endowment vector e and the liability matrix L, we will simply refer to $\mathcal{D}$ as a Γ-solution instead of a Γ-solution to (e, L). Furthermore, for the sake of simplicity, a solution set $\mathcal{D}$ will be said to satisfy axioms in Γ. This is a slight abuse of notation, as a more formal statement should be that a CPM X exists satisfying axioms in Γ such that the implementation of X leads to the set of defaulting firms $\mathcal{D}$.

A solution set $\mathcal{D}$ will be said to be *minimal* if it is the smallest solution in the sense of inclusion. ~~A minimal set $\mathcal{D}$ is a subset of any other solution set.~~ Henceforth, we will simply refer to a minimal solution set as a *minimal default set*.

In the remainder of the paper, we will focus on minimal solutions with the implicit objective of designing a CPM that yields the smallest set of defaulting firms.

3 Results

We now state our results. In the remainder of the paper, we consider as given a vector of initial endowments $e \in \mathcal{E}$, a liability matrix $L \in \mathcal{L}$, and the two parameters driving default cost, $\alpha, \beta \in [0, 1]$. The two last parameters will be further specified for some results.

3.1 Results of Eisenberg and Noe (2001) and Rogers and Veraart (2013)

We start by expressing the main of results of Eisenberg and Noe (2001) and Rogers and Veraart (2013); however, unlike these initial papers, we use minimal default sets rather than CPMs. The following proposition formalizes these results and develops them by providing a slight extension of the initial results.[4]

Proposition 1

The set of {LL,AP,P,LP}-solutions to (e, L) is not empty. Moreover, if $\mathcal{D} \subseteq \mathcal{N}$ is a minimal {LL,AP,P,LP}-solution, then:

1. *$\mathcal{D}_0(e, L) = \emptyset$ if and only if $\mathcal{D} = \emptyset$,*

2. *$\mathcal{D}_0(e, L) \subseteq \mathcal{D}$.*

The first part of Proposition 1 confirms the existence of a minimal default set when the four axioms hold. This is a direct consequence of Rogers and Veraart (2013). Point (1) states that there are no level-0-defaulting firms if and only if we can find a CPM that yields a zero default. The key point to note regarding this statement is that, in this case, the nominal liability matrix L can be directly implemented as a CPM since it verifies the four axioms and, by construction, yields no default. Finally, Point (2) of Proposition 1 explains that level-0-defaulting firms always belong to the minimal default set. No matter

[4] All formal proofs are presented in the Appendix.

which CPM is chosen, level-0-defaulting firms will always end up defaulting, and rescuing them is not feasible. This statement is a slight extension of Rogers and Veraart (2013).

A final remark regarding the {LL,AP,P,LP} axioms is that the *fictitious default algorithm* described in Rogers and Veraart (2013, Section 3.2) can be used to obtain an explicit characterization of the minimal {LL,AP,P,LP}-solution, which corresponds to the greatest clearing payment vector. Although their algorithm focuses on CPMs, it is also useful for determining firm default sets.

3.2 Relaxing the Absolute Priority axiom

We now study the impact of relaxing the Absolute Priority axiom on the minimal defaulting set. The result is summarized in the following proposition.

Proposition 2

The set $\mathcal{D} \subseteq \mathcal{N}$ is a minimal {LP,LL,P,AP}-solution if and only if $\mathcal{D}$ is a minimal {LP,LL,P}-solution.

Proposition 2 states that Absolute Priority is redundant when we focus on the minimal default set. In other words, any defaulting firm will end up with zero worth, even if it is not explicitly imposed. The intuition underlying Proposition 2 is quite simple. Consider a CPM that violates Absolute Priority. This means that a defaulting firm is allowed to end up with positive net worth. These resources come at the expense of other firms (because of default) and reduce those firms' ability to satisfy their creditor obligations. This therefore fosters contagion and is likely to yield a larger minimal default set.

Interestingly, the result in Proposition 2 only holds for minimal default sets. It does not extend to any admissible CPM or to arbitrary – and therefore non-minimal – solution sets. An arbitrary admissible CPM will in general yield a non-minimal solution set, for which the result in Proposition 2 does not hold. For Proposition 2 to hold, the CPM must correspond to the "most favorable" outcome of a minimal default set. ~~This result therefore relies on~~

~~our concept of minimal solution sets and could hold in neither Eisenberg and Noe (2001) (with $\alpha = \beta = 1$) nor in Rogers and Veraart (2013).~~ Obviously, the violation of Proposition 2 only concerns the *if* part of the proposition, since the *only if* part holds for any solution set (and for any CPM). Example 1 below shows that Proposition 2 does not hold for non-minimal solutions.

Example 1 Let $N = 2$, $\alpha > 0$, $e = (2/\alpha, 2/\alpha)$ and $L = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We consider the CPM $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, which can be shown to satisfy LP, P, and LL. In that case, there is a solution set $\mathcal{D}(L, X) = \{2\}$ and $\{2\}$ is an {LP,LL,P}-solution. However, we can prove that $\{2\}$ is not an {LP,LL,P,AP}-solution. Let $X' = \begin{pmatrix} 0 & X'_{12} \\ X'_{21} & 0 \end{pmatrix}$ be a CPM satisfying LP, LL, P, and AP that corresponds to $\mathcal{D}(L, X') = \{2\}$. Because of Proportionality and Absolute Priority, we have $X'_{12} = 1$ and $2 + \beta = X'_{21}$. However, since $\beta \geq 0$, X' does not satisfy Limited Payments. We deduce that $\{2\}$ is not an {LP,LL,P,AP}-solution and that Proposition 2 does not hold for arbitrary solutions.

There is no such issue with the minimal default set. The liability matrix L can be implemented as a CPM and the minimal default set is characterized by no default. Formally, the minimal {LP,LL,P,AP}- and {LP,LL,P}-solutions are both the empty set. ■

We can further characterize the minimal default set for {LP,LL,P}-solutions that can be shown to be unique. The next proposition formalizes this result.

Proposition 3

The set of minimal {LP,LL,P}-solutions contains exactly one element.

The uniqueness result of Proposition 3 simplifies the interpretation of minimal default sets and is intuitive. With this result, minimal default sets can ~~to some extent~~ be seen as *the* "best-case" scenario, in which the number of defaulting firms is minimized. From

Rogers and Veraart (2013) – and the correspondence between minimal default set and greatest clearing payment vector – we further know that these defaulting firms have the greatest possible equity. In particular, Proposition 3 rules out the existence of two distinct minimal default sets (for the same pair (e, L)). In this case, since the inclusion only defines a partial order on $\mathcal{N}$, the two distinct sets would be non-comparable and they could also have different cardinals. Interpreting the minimal set would have been harder and less intuitive.

3.3 Relaxing the Limited Payments axiom

3.3.1 A first implication.

Now that the Absolute Priority axiom has been shown to be redundant, we turn to the Limited Payments axiom. Proposition 4 states that in the absence of default costs, the Limited Payments axiom plays a similar, though more subtle, role than Absolute Priority.

Proposition 4

Let $\alpha = \beta = 1$. If $\mathcal{D} \subseteq \mathcal{N}$ is a minimal {LP,LL,P}-solution, then $\mathcal{D}$ is also a minimal {LL,P}-solution.

As a preliminary remark, it should be noted that Proposition 4 is a one-sided implication, while Proposition 2 is an equivalence. As we explain further in Proposition 5 below, a minimal {LL,P}-solution is not necessarily a minimal {LP,LL,P}-solution.

Now that we have stated what is excluded from Proposition 4, we clarify its meaning. We consider a minimal {LP,LL,P}-solution $\mathcal{D} \subseteq \mathcal{N}$, with no default cost ($\alpha = \beta = 1$). By definition, this means that a CPM X exists that satisfies P, LP, and LL such that $\mathcal{D}(L, X) = \mathcal{D}$. Obviously, $\mathcal{D}$ is also an {LL,P}-solution. However, since the Limited Payments axiom has been removed, there are fewer constraints on the CPM. It could therefore be possible that another CPM $X' \in \mathcal{L}$ exists satisfying only LL and P, such that $\mathcal{D}(L, X') \subsetneq \mathcal{D}$.[5] The existence of such an X' is precisely prevented by Proposition 4. Although removing Limited Payments does not allow us to shrink the minimal solution set, the axiom cannot be said to be redundant, as in the Absolute Priority case.

Removing Limited Payments has strong positive implications. It means that firms may pay more than their contractual obligations. Allowing this to happen may even contradict the very notion of a clearing market, where clearing is understood in a "decentralized" sense. The absence of Limited Payments means that the market can be interpreted as a centralized clearing system, in which a central entity is entitled to tax the assets of firms with positive net worth to absorb the losses of firms with negative net worth. This transfer of resources from positive- to negative-worth firms rules out any pre-committed agreement.

For this reason, Proposition 4 should be understood as a normative statement. It explains that, in the absence of default costs, the close-to-centralized resolution of systemic defaults in financial networks yields better (in the sense of a smaller default set) solutions than clearing markets, where debt contracts are resolved as standard contractual obligations.

Interestingly, Proposition 4 draws a clear line between the presence or the absence of default costs. The Limited Payments axiom is not redundant in the presence of default costs (α or β strictly smaller than 1). In that case, the minimal solution set may potentially be smaller (in the inclusion sense) when the Limited Payments axiom is removed. This is illustrated in Example 2.

Example 2 Let $\alpha = \beta = 1/2$, $N = 4$, $e = (1, 3, 8, 9)$, and $L = \begin{pmatrix} 0 & 6 & 1 & 0 \\ 6 & 0 & 9 & 7 \\ 10 & 8 & 0 & 3 \\ 0 & 4 & 3 & 0 \end{pmatrix}$.

We consider the CPM $X = \begin{pmatrix} 0 & 51/7 & 17/14 & 0 \\ 6 & 0 & 9 & 7 \\ 10 & 8 & 0 & 3 \\ 0 & 4 & 3 & 0 \end{pmatrix}$. It is straightforward to

[5]The notation $A \subsetneq B$ means that A is a proper subset of B.

check that X satisfies P, LL (and AP) and that it corresponds to $\mathcal{D}(L, X) = \{1\}$. Since the empty set – the absence of default – is not an {LL,P}-solution, the singleton $\{1\}$ is a minimal {LL,P}-solution.

We now show that adding the Limited Payments axiom yields a larger minimal solution set. We consider a CPM $X^{(1)} = \begin{pmatrix} 0 & 6x_1^{(1)} & x_1^{(1)} & 0 \\ 6x_2^{(1)} & 0 & 9x_2^{(1)} & 7x_2^{(1)} \\ 10x_3^{(1)} & 8x_3^{(1)} & 0 & 3 \\ 0 & 4 & 3 & 0 \end{pmatrix}$ with $x_1^{(1)} = 3353/11162$, $x_2^{(1)} = 1685/11162$, and $x_3^{(1)} = 2567/11162$ – the form of $X^{(1)}$ comes from the Proportionality axiom. Our calculations show that $X^{(1)}$ satisfies the axioms LP, P, LL (and AP) and we obtain the solution set $\mathcal{D}(L, X^{(1)}) = \{1, 2, 3\}$.[6] We demonstrate that this solution set is minimal by showing that smaller sets cannot be solutions. Taken together with our previous statement that $\{1\}$ is a minimal {LL,P}-solution, this concludes our counter-example.

We can check that firm 2 is the sole level-0-defaulting firm. Point (2) of Proposition 1 implies that $\{1, 3\}$ is not an {LP,P,LL,AP}-solution and, hence, is not a minimal {LP,P,LL}-solution. As a result of Proposition 2, $\{1, 3\}$ is therefore not a minimal {LP,P,LL}-solution.

Let us assume that $\{1, 2\}$ is a minimal {LP,P,LL}-solution. There is then a CPM $X^{(2)} = \begin{pmatrix} 0 & 6x_1^{(2)} & x_1^{(2)} & 0 \\ 6x_2^{(2)} & 0 & 9x_2^{(2)} & 7x_2^{(2)} \\ 10 & 8 & 0 & 3 \\ 0 & 4 & 3 & 0 \end{pmatrix}$ satisfying LP, P, and LL that corresponds to the minimal solution set $\mathcal{D}(L, X^{(2)}) = \{1, 2\}$. By applying Proportionality and the definition of $\mathcal{D}(L, X^{(2)})$, we must have $x_1^{(2)}, x_2^{(2)} < 1$. This implies that $x_1^{(2)} + 9x_2^{(2)} - 10 < 0$, while Limited Liability for firm 3

[6] An alternative solution to construct the minimal set would be to use the fictitious default algorithm of Rogers and Veraart (2013). For the paper to remain self-contained, we provide a stand-alone proof.

yields the opposite inequality. Hence, by contradistinction, $\{1, 2\}$ is not an $\{$LP,P,LL$\}$-solution. Analogous reasoning shows that no singleton can be an $\{$LP,P,LL$\}$-solution.

Let us now assume that $\{2, 3\}$ is a minimal $\{$LP,P,LL$\}$-solution. In accordance with Proposition 2, $\{2, 3\}$ is also a minimal $\{$LP,P,LL,AP$\}$-solution. A CPM $X^{(3)} = \begin{pmatrix} 0 & 6 & 1 & 0 \\ 6x_2^{(3)} & 0 & 9x_2^{(3)} & 7x_2^{(3)} \\ 10x_3^{(3)} & 8x_3^{(3)} & 0 & 3x_3^{(3)} \\ 0 & 4 & 3 & 0 \end{pmatrix}$ therefore exists, satisfying LP, P, LL, AP, such that $\mathcal{D}(L, X^{(3)}) = \{2, 3\}$. Because firms 2 and 3 are defaulting, Absolute Priority implies that $\begin{cases} (13 + 8x_3^{(3)})/2 - 22x_2^{(3)} = 0 \\ (12 + 9x_2^{(3)})/2 - 21x_3^{(3)} = 0 \end{cases}$. After computation, we obtain $x_2^{(3)} = 107/296$ and $x_3^{(3)} = 215/592$. Limited Liability for firm 1 implies $6x_2^3 + 10x_2^3 - 6 > 0$, which does not hold with the actual values of $x_2^{(3)}$ and $x_3^{(3)}$, the result being $-118/592 < 0$. The set $\{2, 3\}$ is therefore not a minimal $\{$LP,P,LL$\}$-solution.

No set of cardinal one or two can be a solution, which concludes our proof. ■

Since Proposition 2 states that the Absolute Priority axiom is redundant in case of a $\{$LP,LL,P$\}$-solution, the result of Proposition 4 can equivalently be stated as follows.

Corollary 1

Let $\alpha = \beta = 1$. If $\mathcal{D} \subseteq \mathcal{N}$ is a minimal {LP,LL,P,AP}-solution, then $\mathcal{D}$ is a minimal {LL,P~~,AP~~}-solution.

3.3.2 A partial converse result

As briefly noted, Proposition 4 only presents a one-directional implication, while Proposition 2 regarding Absolute Priority features a result based on an equivalence. In short, Absolute Priority is a redundant axiom: solutions with or without the axiom are the same. This is not the case with Limited Payments, even in the absence of default costs.

In particular, a minimal solution in the absence of Absolute Priority is not necessarily a minimal solution when Absolute Priority is imposed. Proposition 4 only states that a minimal solution with Absolute Priority is also a solution without this axiom. More formally, Proposition 5 below shows that the converse of Proposition 4 – and of Corollary 1 – does not hold.

Proposition 5

Let $\alpha = \beta = 1$.

1. *Let $\mathcal{D} \in \mathcal{N}$ be a minimal {LL,P}-solution. $\mathcal{D}$ is not necessarily a minimal {LP,LL,P}-solution.*

2. *Let $\mathcal{D} \in \mathcal{N}$ be a minimal {LL,P,AP}-solution. $\mathcal{D}$ is not necessarily a minimal {LP,LL,P}-solution.*

Note that the second point of Proposition 5 stems directly from the first point, in accordance with Proposition 2. Proposition 5 states that the possibility of taxing wealthier firms to finance the liabilities of poorer firms exists in the absence of Limited Payments. Some minimal {LL,P}-solution sets take advantage of the relaxation of the Absolute Priority axiom, although Proposition 4 makes it clear that this is not the case for *all* {LL,P}-solutions. In Example 3, we prove Proposition 5 by providing an explicit illustration.

In order to prove Proposition 5, we consider the following example.

Example 3 Let $N = 4$, $e = (1, 2, 2, 9)$, and $L = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

We define the CPM $X^{(1)} = \begin{pmatrix} 0 & 5 & 0 & 0 \\ 2 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, which can be shown to satisfy LL and P. It is straightforward to check that $\mathcal{D}(L, X^{(1)}) = \{1\}$. Moreover, we can verify that the empty set – no default – cannot be a solution. Thus, $\{1\}$ is a minimal {LL,P}-solution.

We now check that there is no $X \in \mathcal{L}$ satisfying LP, P, LL with $\mathcal{D}(L, X) = \{1\}$. Consider such a matrix X. Because of Limited Liability, we have $X_{\mathcal{N},2} + e_2 - X_{2,\mathcal{N}} \geq 0$, which becomes $1 + X_{12} + 2 - 5 \geq 0$ because of Proportionality. This inequality cannot hold since the Limited Payments axiom requires that $X_{12} \leq 1$.

Of note, the CPM $X^{(2)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 22/14 & 0 & 11/14 & 22/14 \\ 13/14 & 13/14 & 0 & 13/14 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ corresponding to a minimal {LL,P,LP}-solution is $\{2, 3\}$. This solution is unique, in accordance with Eisenberg and Noe (2001). ■

3.4 Forcing level-0-defaulting firms to default

As can be seen in Example 3, relaxing the Limited Payments axiom may mean that a firm not initially in $\mathcal{D}_0(e, L)$ (such as firm 1 in Example 3) is "sacrificed" in order to rescue a level-0-defaulting firm in $\mathcal{D}_0(e, L)$ (firm 2 in this example). This may be a normative disputable consequence of relaxing the Limited Payments axiom. In order to prevent such situations occurring, we introduce the Inclusion axiom below.

Axiom 5 (Inclusion of Level-0-Defaulting Firms, I)
A CPM $X \in \mathcal{L}$ satisfies the Inclusion of Level-0-Defaulting Firms if $\mathcal{D}_0(e, L) \subseteq \mathcal{D}(L, X)$.

The above axiom, which we will simply refer to as the Inclusion axiom, states that a level-0-defaulting firm should not be rescued and should belong to the default solution set. This axiom rules out the situation shown in Example 3, for instance.

Our first result is a direct corollary of Proposition 4 and of the second point of Proposition 1. Any minimal {LP,LL,P}-solution contains all level-0-defaulting firms.

Corollary 2

Let $\alpha = \beta = 1$. If $\mathcal{D} \in \mathcal{N}$ is a minimal {LP,LL,P}-solution, then $\mathcal{D}$ is a minimal {LL,P,I}-solution.

However, the following proposition makes it clear that Corollary 2 is the only positive result with respect to the Inclusion axiom.

Proposition 6

Let $\alpha = \beta = 1$.

1. *Let $\mathcal{D} \in \mathcal{N}$ be a minimal {LL,P,I}-solution.*
   - *(a) $\mathcal{D}$ is not necessarily a minimal {LL,P}-solution.*
   - *(b) $\mathcal{D}$ is not necessarily a minimal {LP,LL,P}-solution.*
2. *Let $\mathcal{D} \in \mathcal{N}$ be a minimal {LL,P}-solution. $\mathcal{D}$ is not necessarily a minimal {LL,P,I}-solution.*

The first point of Proposition 6 explains that adding the Inclusion axiom does not allow us to overcome the result of Proposition 5. It could be the case that removing the Inclusion axiom causes the minimal solution set to shrink. The second point explains that a minimal {LL,P}-solution may violate the Inclusion axiom.

Implications for clearing houses. Our results echo the case of Central Counterparty Clearing Houses (CCPs), considered, for instance, in Eisenberg and Noe (2001). Actual CCPs often manage the default of one of their members according to some pre-committed "waterfall" process (see for instance ISDA 2013 for a detailed description of the waterfall process). The first two stages of the waterfall involve the defaulting member only, but if needed, the deposits of other members may be used to avoid the collapse of the CCP. In such a circumstance, non-defaulting members are required to *pay more* than their initial liabilities. Through the lenses of our axioms, these higher payments can be viewed as a violation of the Limited Payments axiom. As Proposition 4 shows, this has no effect on the set of default firms if there is no default cost, i.e., if both α and β are equal to one. Actual CCPs now impose haircuts to liabilities in the event of default. As shown in Example 2, violations of the Limited Payments axiom may then be used to rescue other members.

4 Discussion

Our axiomatic analysis of the Eisenberg-Noe framework relies on four initial axioms: *Limited Liability*, *Absolute Priority*, *Proportionality*, and *Limited Payments*. Investigating the consequences of these axioms on the minimal sets of defaulting firms yields two main results. First, the Absolute Priority axiom is redundant. Adding or removing it leaves the minimal set unchanged. Second, the Limited Payments axiom is similar, although more subtle. In the absence of default costs, we cannot shrink the minimal default set by relaxing the axiom, while imposing it may lead to larger (in the sense of inclusion) sets. In addition, we provide several counterexamples showing that no other similar result can be derived by further weakening the axioms.

Our novel approach based on minimal default sets allows us to draw new normative conclusions. As already stated in the introduction, minimal default sets are linked to macro-prudential regulation and could complement existing tools for identifying G-SIFIs by offering a new and complementary perspective. We are aware, however, that our approach also has certain limitations. One of these is that our results are only indirectly related to the CPM, which reduces their operational usability. Our study should therefore be viewed as a normative contribution rather than as a positive one.

Finally, we believe that our study could be extended in two main directions. First, we mainly rely on the axioms of Eisenberg and Noe (2001); however, it would be interesting to analyze the introduction of other – and weaker – axioms. As an example, the Proportionality axiom we use implies that all claims are equally privileged. Introducing seniority rules, as in Elsinger et al. (2006), could be a fruitful avenue of investigation. Second, we only compare default sets based on the inclusion ranking. However, some sets might present appealing features if a richer order were to be used, one that accounted for firm size, for instance. We also leave this question for future research.

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Appendix

A Proof of Proposition 1

Let $0 \leq \alpha, \beta \leq 1$, $e = (e_j)_{j \in \mathcal{N}} \in \mathcal{E}$, and $L = (L_{ij})_{i,j \in \mathcal{N}} \in \mathcal{L}$. The set of {LL,AP,P,LP}-solutions to (e, L) that are non-empty is a result of Rogers and Veraart (2013).

A.1 Proof of Proposition 1-1

If: Let us assume $\mathcal{D}_0(e, L) = \emptyset$. The proof is straightforward by showing that L satisfies LL, AP, P, and LP.

Only if: Let us assume that the empty set is an {LL,AP,P,LP}-solution to (e, L). There is therefore a CPM $X \in \mathcal{L}$ such that $\mathcal{D}(L, X) = \emptyset$. By definition, $X = L$ and $\mathcal{D}_0(e, L) = \emptyset$ therefore follow from LL.

A.2 Proof of Proposition 1-2

Let $X = (X_{ij})_{i,j \in \mathcal{N}} \in \mathcal{L}$ be a CPM satisfying LL, AP, P, and LP such that $\mathcal{D}(L, X)$ is a minimal {LL,AP,P,LP}-solution to (e, L). Let us assume that we can consider $k \in \mathcal{N}$ such that $k \in \mathcal{D}_0(e, L)$ and $k \notin \mathcal{D}(L, X)$.

Because X satisfies LL, we have $e_k + X_{\mathcal{N},k} - X_{k,\mathcal{N}} \geq 0$. Because X satisfies P, we have $\forall i$, such that $L_{i,\mathcal{N}} > 0$, $X_{i,k} = L_{i,k} \dfrac{X_{i,\mathcal{N}}}{L_{i,\mathcal{N}}} \leq L_{i,k}$, where the inequality holds since X also verifies LP. The P axiom also implies that $\forall i$, such that $L_{i,\mathcal{N}} = 0$, $X_{i,k} = 0$. We deduce by summing the inequalities over i that: $X_{\mathcal{N},k} \leq L_{\mathcal{N},k}$. By definition of $\mathcal{D}(L, X)$, $X_{k,\mathcal{N}} = L_{k,\mathcal{N}}$. Combining the latter with the previous inequality gives:

$$e_k + L_{\mathcal{N},k} - L_{k,\mathcal{N}} \geq 0,$$

which contradicts the assumption that $k \in \mathcal{D}_0(e, L)$.

B Proof of Proposition 2

B.1 A preliminary lemma

Lemma 1

Let $0 \leq \alpha, \beta \leq 1$. Let $e = (e_j)_{j \in \mathcal{N}} \in \mathcal{E}$ and $L = (L_{ij})_{i,j \in \mathcal{N}} \in \mathcal{L}$. Let the CPM $X \in \mathcal{L}$ satisfying LP, LL, P be such that $\mathcal{D}(L, X)$ is a minimal {LP,LL,P}-solution to (e, L). There is therefore a CPM $X' \in \mathcal{L}$ satisfying LP, LL, P, AP such that $\mathcal{D}(L, X') = \mathcal{D}(L, X)$.

Proof of Lemma 1: Let $X = (X_{ij})_{i,j \in \mathcal{N}}$ satisfy LP, LL, P such that $\mathcal{D}(L, X)$ is a minimal {LP,LL,P}-solution to (e, L). If $\mathcal{D}(L, X) = \emptyset$, AP is trivially satisfied and the lemma is proved with $X' = X$. Then, assume $\mathcal{D}(L, X) \neq \emptyset$. In the remainder of the proof, we use the following notation: $\mathcal{D} = \mathcal{D}(L, X)$ and $\overline{\mathcal{D}} = \overline{\mathcal{D}}(L, X)$. Since X satisfies LP,

$$\forall i \in \mathcal{N}, X_{i,\mathcal{N}} \leq L_{i,\mathcal{N}}. \tag{2}$$

Moreover, because X satisfies LL and P, we can use $\mathcal{N} = \mathcal{D} \cup \overline{\mathcal{D}}$ to obtain:

$$\begin{aligned} \forall i \in \mathcal{D}, X_{i,\mathcal{N}} - \beta.\sum_{j \in \mathcal{D}} \frac{L_{ji}}{L_{j,\mathcal{N}}} X_{j,\mathcal{N}} &\leq \alpha.e_i + \beta.\sum_{j \in \overline{\mathcal{D}}} L_{ji}, \\ \forall i \in \overline{\mathcal{D}}, L_{i,\mathcal{N}} - \sum_{j \in \mathcal{D}} \frac{L_{ji}}{L_{j,\mathcal{N}}} X_{j,\mathcal{N}} &\leq e_i + \sum_{j \in \overline{\mathcal{D}}} L_{ji}. \end{aligned} \tag{3}$$

To prove the lemma and find the X' we are looking for, we start by characterizing this CPM on the set of defaulting firms. We can easily deduce the CPM for non-defaulting firms, as these firms pay their liabilities in full. We consider the following linear program:

$$T = \arg \max_{(X'_{i,\mathcal{N}})_{i \in \mathcal{D}}} \sum_{i \in \mathcal{D}} X'_{i,\mathcal{N}} \tag{4}$$

subject to:

$$\left\{ \begin{array}{l} \forall i \in \mathcal{D},\ 0 \leq X'_{i,\mathcal{N}} \leq L_{i,\mathcal{N}}, \\ \forall i \in \mathcal{D},\ X'_{i,\mathcal{N}} - \beta.\sum_{j \in \mathcal{D}} \frac{L_{ji}}{L_{j,\mathcal{N}}} X'_{j,\mathcal{N}} \leq \alpha.e_i + \beta.\sum_{j \in \overline{\mathcal{D}}} L_{ji}, \\ \forall i \in \overline{\mathcal{D}}, \hat{A}\ \ L_{i,\mathcal{N}} - \sum_{j \in \mathcal{D}} \frac{L_{ji}}{L_{j,\mathcal{N}}} X'_{j,\mathcal{N}} \leq e_i + \sum_{j \in \overline{\mathcal{D}}} L_{ji}. \end{array} \right. \tag{5}$$

The constraints in Equation (5) of the program (4)–(5) are linear and define a compact set. Hence, the program (4)–(5) has a non-empty set of solutions. Let $\overline{X} = (\overline{X}_{i,\mathcal{N}})_{i\in\mathcal{D}}$ be such a solution.

We now extend this definition from $\mathcal{D}$ to $\mathcal{N}$. Let $\overline{X^N} = (\overline{X^N_{ij}})_{i,j\in\mathcal{N}}$ be defined by $\forall i \in \mathcal{D}$, $\overline{X^N_{ij}} = \overline{X}_{i,\mathcal{N}} \dfrac{L_{ij}}{L_{i,\mathcal{N}}}$, and $\forall i \in \overline{\mathcal{D}}$, $\overline{X^N_{ij}} = L_{ij}$.

We can check that $\forall i \in \mathcal{D}$, $\overline{X}_{i,\mathcal{N}} = L_{i,\mathcal{N}}$ or $\overline{X}_{i,\mathcal{N}} - \beta \sum_{j\in\mathcal{D}} \dfrac{L_{ji}}{L_{j,\mathcal{N}}} \overline{X}_{j,\mathcal{N}} = \alpha e_i + \beta \sum_{j\in\overline{\mathcal{D}}} L_{ji}$ (otherwise, increasing $\overline{X}_{i,\mathcal{N}}$ by a quantity small enough to ensure that the constraints of Equation (5) are still satisfied is possible, contradicting the assumption that $\overline{X}$ is a solution to program (4)–(5)). If $\exists i \in \mathcal{D}, \overline{X}_{i,\mathcal{N}} = L_{i,\mathcal{N}}$, then $\overline{X^N}$ is such that $\mathcal{D}(L, \overline{X^N}) \subsetneq \mathcal{D}$ and the constraints of Equation (5) imply that $\overline{X^N}$ satisfies LP, P, and LL.[7] This contradicts the fact that $\mathcal{D}$ is a minimal {LP,LL,P}-solution to (e, L). We must therefore have $\forall i \in \mathcal{D}$, $\overline{X}_{i,\mathcal{N}} - \beta \sum_{j\in\mathcal{D}} \dfrac{L_{ji}}{L_{j,\mathcal{N}}} \overline{X}_{j,\mathcal{N}} = \alpha e_i + \beta \sum_{j\in\overline{\mathcal{D}}} L_{ji}$. This implies that $\overline{X^N}$ satisfies AP and the lemma is proved with $X' = \overline{X^N}$. □

B.2 Proof of Proposition 2

Let $0 \leq \alpha, \beta \leq 1$. Let $e = (e_j)_{j\in\mathcal{N}} \in \mathcal{E}$ and $L = (L_{ij})_{i,j\in\mathcal{N}} \in \mathcal{L}$.

1) Let $\mathcal{D}$ be a minimal {LP,LL,P,AP}-solution to (e, L). Let us show that $\mathcal{D}$ is a minimal {LP,LL,P}-solution to (e, L).

a) Let us show that $\mathcal{D}$ is an {LP,LL,P}-solution to (e, L). By definition, a CPM $X \in \mathcal{L}$ exists such that:

- X satisfies LP, LL, P, AP,
- $\mathcal{D} = \mathcal{D}(L, X)$.

[7]Since $0 \leq \alpha, \beta \leq 1$, if $L_{i,\mathcal{N}} - \beta \sum_{j\in\mathcal{D}} \dfrac{L_{ji}}{L_{j,\mathcal{N}}} \overline{X}_{j,\mathcal{N}} \leq \alpha e_i + \beta \sum_{j\in\overline{\mathcal{D}}} L_{ji}$ then, $L_{i,\mathcal{N}} - \sum_{j\in\mathcal{D}} \dfrac{L_{ji}}{L_{j,\mathcal{N}}} \overline{X}_{j,\mathcal{N}} \leq e_i + \sum_{j\in\overline{\mathcal{D}}} L_{ji}$.

Then, by definition, $\mathcal{D}$ is an {LP,LL,P}-solution to (e, L).

b) Let us show that $\mathcal{D}$ is a *minimal* {LP,LL,P}-solution to (e, L). In order to obtain a contradiction, let $\mathcal{D}' \subsetneq \mathcal{D}$ be an {LP,LL,P}-solution to (e, L). The CPM $X \in \mathcal{L}$ then exists such that:

- X satisfies LP, LL, P,
- $\mathcal{D}' = \mathcal{D}(L, X)$.

Then, by Lemma 1, there is a CPM $X' \in \mathcal{L}$ such that:

- X' satisfies LP, LL, P, AP,
- $\mathcal{D}' = \mathcal{D}(L, X')$.

Then, $\mathcal{D}'$ is an {LP,LL,P,AP}-solution to (e, L), contradicting the assumption that $\mathcal{D}$ is a minimal {LP,LL,P,AP}-solution to (e, L).

2) Let $\mathcal{D}$ be a minimal {LP,LL,P}-solution to (e, L). Let us show that $\mathcal{D}$ is a minimal {LP,LL,P,AP}-solution to (e, L).

a) Let us show that $\mathcal{D}$ is an {LP,LL,P,AP}-solution to (e, L). As $\mathcal{D}$ is a minimal {LP,LL,P}-solution to (e, L), this implies, by definition, that a CPM $X \in \mathcal{L}$ exists such that:

- X satisfies LP, LL, P,
- $\mathcal{D} = \mathcal{D}(L, X)$.

Then, from Lemma 1, there is a CPM $X' \in \mathcal{L}$ satisfying LP, LL, P, AP such that $\mathcal{D}(L, X') = \mathcal{D}(L, X)$. Hence, by definition, $\mathcal{D}$ is an {LP,LL,P,AP}-solution to (e, L).

b) Let us show that $\mathcal{D}$ is a *minimal* {LP,LL,P,AP}-solution to (e, L). Conversely, let $\mathcal{D}' \subsetneq \mathcal{D}$ be an {LP,LL,P,AP}-solution to (e, L). By definition, there is a CPM $X \in \mathcal{L}$ such that:

- X satisfies LP, LL, P, AP,

- $\mathcal{D}' = \mathcal{D}(L, X)$.

Then, by definition, $\mathcal{D}'$ is an {LP,LL,P}-solution to (e, L), contradicting the fact that $\mathcal{D}$ is a minimal {LP,LL,P}-solution to (e, L).

C Proof of Proposition 3

Let $0 \leq \alpha, \beta \leq 1$. Let $e = (e_j)_{j\in\mathcal{N}} \in \mathcal{E}$ and $L = (L_{i,j})_{i,j\in\mathcal{N}} \in \mathcal{L}$.

The proposition that the set of minimal {LL,P,LP}-solutions to (e, L) is non-empty is a straightforward corollary of Rogers and Veraart (2013), Proposition 2, and the definition of minimality. Let us prove the uniqueness of the minimal {LL,P,LP}-solution to (e, L).

Let us assume, on the contrary, that there are two CPMs, $X^1 = (X^1_{ij})_{i,j\in\mathcal{N}}$ and $X^2 = (X^2_{ij})_{i,j\in\mathcal{N}}$, satisfying LL, P, and LP such that $\mathcal{D}(L, X^1)$ and $\mathcal{D}(L, X^2)$ are minimal {LL,P,LP}-solutions to (e, L) and $\mathcal{D}(L, X^1) \neq \mathcal{D}(L, X^2)$. Let us define $D^0 = \mathcal{D}(L, X^1) \cap \mathcal{D}(L, X^2)$, $D^1 = \mathcal{D}(L, X^1) \backslash \mathcal{D}(L, X^2)$, $D^2 = \mathcal{D}(L, X^2) \backslash \mathcal{D}(L, X^1)$, and $\overline{N} = N \backslash (\mathcal{D}(L, X^1) \cup \mathcal{D}(L, X^2))$. From the definition of minimality, we must have $D^1 \neq \emptyset$ and $D^2 \neq \emptyset$.

Let us define the CPM $X^M = (X^M_{ij})_{i,j\in\mathcal{N}}$ as $\forall i, j \in \mathcal{N}, X^M_{ij} = \max\left(X^1_{ij}, X^2_{ij}\right)$.

From the definition of defaulting firms and from LP, we have $\mathcal{D}(L, X^M) = D^0$.

Because X^1 and X^2 satisfy P, it is straightforward to check that X^M satisfies P. It is also straightforward to check that when X^1 and X^2 satisfy LP then X^M also satisfies LP.

Let us show that X^M satisfies LL.

1. Let $i \in D^0$. When X^1 and X^2 satisfy LL, this implies:

$$\beta\left(X^1_{D^0,i} + X^1_{D^1,i} + X^1_{D^2,i} + X^1_{\overline{N},i}\right) + \alpha e_i \geq X^1_{i,\mathcal{N}},$$

and

$$\beta\left(X^2_{D^0,i} + X^2_{D^1,i} + X^2_{D^2,i} + X^2_{\overline{N},i}\right) + \alpha e_i \geq X^2_{i,\mathcal{N}}.$$

By definition of X^M and P, we have $X^M_{D^0,i} \geq X^1_{D^0,i}$ and $X^M_{D^0,i} \geq X^2_{D^0,i}$, $X^M_{D^1,i} = X^2_{D^1,i} = L_{D^1,i} > X^1_{D^1,i}$, $X^M_{D^2,i} = X^1_{D^2,i} = L_{D^2,i} > X^2_{D^2,i}$, and $X^M_{\overline{N},i} = X^1_{\overline{N},i} = X^2_{\overline{N},i} = L_{\overline{N},i}$. Also, from P, we have $X^M_{i,\mathcal{N}} = X^1_{i,\mathcal{N}}$ or $X^M_{i,\mathcal{N}} = X^2_{i,\mathcal{N}}$. Hence,

$$\beta \left(X^M_{D^0,i} + X^M_{D^1,i} + X^M_{D^2,i} + X^M_{\overline{N},i} \right) + \alpha e_i \geq X^M_{i,\mathcal{N}}.$$

2. Let $i \in D^1$. X^2 satisfying LL implies:

$$\beta \left(X^2_{D^0,i} + X^2_{D^1,i} + X^2_{D^2,i} + X^2_{\overline{N},i} \right) + \alpha e_i \geq X^2_{i,\mathcal{N}}.$$

By definition of X^M and P, we have $X^M_{D^0,i} \geq X^2_{D^0,i}$, $X^M_{D^1,i} = X^2_{D^1,i} = L_{D^1,i}$, $X^M_{D^2,i} > X^2_{D^2,i}$, $X^M_{\overline{N},i} = X^2_{\overline{N},i} = L_{\overline{N},i}$, and $X^2_{i,\mathcal{N}} = X^M_{i,\mathcal{N}} = L_{i,\mathcal{N}}$. Hence,

$$\beta \left(X^M_{D^0,i} + X^M_{D^1,i} + X^M_{D^2,i} + X^M_{\overline{N},i} \right) + \alpha e_i \geq X^M_{i,\mathcal{N}}.$$

The proof for $i \in D^2$ and $i \in \overline{N}$ is identical and is not presented here.

We have shown that X^M also satisfies LL.

$\mathcal{D}(L, X^M)$ is then strictly included in $\mathcal{D}(L, X^1)$ and $\mathcal{D}(L, X^2)$ and X^M satisfies LL, P, and LP. There is hence a contradiction with $\mathcal{D}(L, X^1) \neq \mathcal{D}(L, X^2)$ being minimal {LL,P,LP}-solutions to (e, L).

D Proof of Proposition 4

Let $\alpha = \beta = 1$. Let $e = (e_j)_{j\in\mathcal{N}} \in \mathcal{E}$ and $L = (L_{ij})_{i,j\in\mathcal{N}} \in \mathcal{L}$. Let $\mathcal{D}^1 \subseteq \mathcal{N}$ be a minimal {LP,LL,P}-solution to (e, L). From Proposition 2, $\mathcal{D}^1$ is a minimal {LP,LL,P,AP}-solution to (e, L). Assume, in contradiction with this proposition, that $\mathcal{D}^2 \subsetneq \mathcal{D}^1$ is an {LL,P}-solution to (e, L).

If $\mathcal{D}^2 = \emptyset$, then LP is trivially satisfied and $\mathcal{D}^2$ is an {LP,LL,P}-solution to (e, L), contradicting the assumption that $\mathcal{D}^1$ is a minimal {LP,LL,P}-solution to (e, L). Now let us assume that $\mathcal{D}^2 \neq \emptyset$. Let $X^1 = (X^1_{ij})_{i,j\in\mathcal{N}} \in \mathcal{L}$ be such that:

- X^1 satisfies LP, LL, P, AP,
- $\mathcal{D}^1 = \mathcal{D}(L, X^1)$.

X^1 exists since, by assumption, $\mathcal{D}^1$ is a minimal {LP,LL,P,AP}-solution to (e, L).

Let $\mathcal{X}$ be the set of clearing payment matrices $X \in \mathcal{L}$ such that:

- X satisfies LL, P,
- $\mathcal{D}^2 = \mathcal{D}(L, X)$.

If we assume that $\mathcal{D}^2$ is an {LP,LL,P}-solution to (e, L), then $\mathcal{X} \neq \emptyset$. For any $X = (X_{ij})_{i,j \in \mathcal{N}} \in \mathcal{X}$, let us define $\mathcal{D}^{2+}(X) = \{i \in \mathcal{D}^2, X_{i,\mathcal{N}} > X^1_{i,\mathcal{N}}\}$, $\mathcal{D}^{2-}(X) = \{i \in \mathcal{D}^2, X_{i,\mathcal{N}} < X^1_{i,\mathcal{N}}\}$, $\mathcal{D}^{2=}(X) = \{i \in \mathcal{D}^2, X_{i,\mathcal{N}} = X^1_{i,\mathcal{N}}\}$. Let us have $X^2 \in \mathcal{X}$ such that $\forall X \in \mathcal{X}, \neg(\mathcal{D}^{2-}(X) \subsetneq \mathcal{D}^{2-}(X^2))$. By definition, $\mathcal{D}^{2+}(X^2) \cup \mathcal{D}^{2-}(X^2) \cup \mathcal{D}^{2=}(X^2) = \mathcal{D}^2$. Moreover, if $\mathcal{D}^{2+}(X^2) = \emptyset$, X^2 obviously satisfies LP which, together with the assumption that X^2 satisfies P and LL, contradicts the assumption that $\mathcal{D}^1$ is a minimal {LP,LL,P}-solution to (e, L). Hence,

$$\mathcal{D}^{2+}(X^2) \neq \emptyset. \tag{6}$$

1) With a proof similar to that of Lemma 1, we can show, with no loss of generality, that X^2 is such that $\forall i \in \mathcal{D}^{2-}(X^2)$, $X^2_{\mathcal{N},i} + e_i = X^2_{i,\mathcal{N}}$.

2) Then, summing over $\mathcal{D}^{2-}(X^2)$, we obtain:

$$\begin{aligned}&L_{N\setminus\mathcal{D}^1,\mathcal{D}^{2-}(X^2)} + L_{\mathcal{D}^1\setminus\mathcal{D}^2,\mathcal{D}^{2-}(X^2)} + \\ &X^2_{\mathcal{D}^{2=}(X^2),\mathcal{D}^{2-}(X^2)} + X^2_{\mathcal{D}^{2+}(X^2),\mathcal{D}^{2-}(X^2)} + e_{\mathcal{D}^{2-}(X^2)} - X^2_{\mathcal{D}^{2-}(X^2),N\setminus\mathcal{D}^{2-}(X^2)} = 0.\end{aligned} \tag{7}$$

Moreover, since X^1 satisfies AP and $\mathcal{D}^{2-}(X^2) \subseteq \mathcal{D}^2 \subsetneq \mathcal{D}^1$,

$$\begin{aligned}&L_{N\setminus\mathcal{D}^1,\mathcal{D}^{2-}(X^2)} + X^1_{\mathcal{D}^1\setminus\mathcal{D}^2,\mathcal{D}^{2-}(X^2)} + \\ &X^1_{\mathcal{D}^{2=}(X^2),\mathcal{D}^{2-}(X^2)} + X^1_{\mathcal{D}^{2+}(X^2),\mathcal{D}^{2-}(X^2)} + e_{\mathcal{D}^{2-}(X^2)} - X^1_{\mathcal{D}^{2-}(X^2),N\setminus\mathcal{D}^{2-}(X^2)} = 0.\end{aligned} \tag{8}$$

Since X^1 and X^2 satisfy P, by definition of $\mathcal{D}^{2=}(X^2)$ and $\mathcal{D}^{2+}(X^2)$, $X^2_{\mathcal{D}^{2=}(X^2),\mathcal{D}^{2-}(X^2)} = X^1_{\mathcal{D}^{2=}(X^2),\mathcal{D}^{2-}(X^2)}$ and $X^2_{\mathcal{D}^{2+}(X^2),\mathcal{D}^{2-}(X^2)} \geq X^1_{\mathcal{D}^{2+}(X^2),\mathcal{D}^{2-}(X^2)}$. Moreover, since X^1 satisfies LP,

$L_{\mathcal{D}_1\setminus\mathcal{D}_2,\mathcal{D}^{2-}(X^2)} \geq X^1_{\mathcal{D}_1\setminus\mathcal{D}_2,\mathcal{D}^{2-}(X^2)}$. This then means that $X^2_{\mathcal{D}^{2-}(X^2),N\setminus\mathcal{D}^{2-}(X^2)} \geq X^1_{\mathcal{D}^{2-}(X^2),N\setminus\mathcal{D}^{2-}(X^2)}$. Together with the definition of $\mathcal{D}^{2-}(X^2)$, we must have $X^2_{\mathcal{D}^{2-}(X^2),N\setminus\mathcal{D}^{2-}(X^2)} = X^1_{\mathcal{D}^{2-}(X^2),N\setminus\mathcal{D}^{2-}(X^2)} = 0$. Hence,

$$\forall i \in \mathcal{D}^{2-}(X^2), \forall j \in \mathcal{N} \setminus \mathcal{D}^{2-}(X^2), X^1_{ij} = X^2_{ij} = 0. \tag{9}$$

From Equations (7) and (8), we have:

$$\forall i \in \mathcal{N} \setminus \mathcal{D}^{2-}(X^2), \forall j \in \mathcal{D}^{2-}(X^2), X^1_{ij} = X^2_{ij} = 0. \tag{10}$$

3) Since X^1 satisfies AP,

$$X^1_{\mathcal{D}^1\setminus\mathcal{D}^2,N\setminus\mathcal{D}^1} + X^1_{\mathcal{D}^{2-}(X^2),N\setminus\mathcal{D}^1} + X^1_{\mathcal{D}^{2=}(X^2),N\setminus\mathcal{D}^1} + X^1_{\mathcal{D}^{2+}(X^2),N\setminus\mathcal{D}^1} - L_{N\setminus\mathcal{D}^1,\mathcal{D}^1} = e_N.$$

Since X^2 satisfies LL,

$$L_{\mathcal{D}^1\setminus\mathcal{D}^2,N\setminus\mathcal{D}^1} + X^2_{\mathcal{D}^{2-}(X^2),N\setminus\mathcal{D}^1} + X^2_{\mathcal{D}^{2=}(X^2),N\setminus\mathcal{D}^1} + X^2_{\mathcal{D}^{2+}(X^2),N\setminus\mathcal{D}^1} - L_{N\setminus\mathcal{D}^1,\mathcal{D}^1} \leq e_N.$$

By definition and the fact that X^1 and X^2 satisfy P, $X^1_{\mathcal{D}^{2=}(X^2),N\setminus\mathcal{D}^1} = X^2_{\mathcal{D}^{2=}(X^2),N\setminus\mathcal{D}^1}$, $X^2_{\mathcal{D}^{2+}(X^2),N\setminus\mathcal{D}^1} \geq X^1_{\mathcal{D}^{2+}(X^2),N\setminus\mathcal{D}^1}$ and since X^1 satisfies LP, $L_{\mathcal{D}^1\setminus\mathcal{D}^2,N\setminus\mathcal{D}^1} \geq X^1_{\mathcal{D}^1\setminus\mathcal{D}^2,N\setminus\mathcal{D}^1}$. Moreover, from Equation (9), $X^2_{\mathcal{D}^{2-}(X^2),N\setminus\mathcal{D}^1} = X^1_{\mathcal{D}^{2-}(X^2),N\setminus\mathcal{D}^1} = 0$. Then,

$$L_{\mathcal{D}^1\setminus\mathcal{D}^2,N\setminus\mathcal{D}^1} + X^2_{\mathcal{D}^{2=}(X^2),N\setminus\mathcal{D}^1} + X^2_{\mathcal{D}^{2+}(X^2),N\setminus\mathcal{D}^1} - L_{N\setminus\mathcal{D}^1,\mathcal{D}^1} = e_N, \tag{11}$$

$$L_{\mathcal{D}^1\setminus\mathcal{D}^2,N\setminus\mathcal{D}^1} = X^1_{\mathcal{D}^1\setminus\mathcal{D}^2,N\setminus\mathcal{D}^1}, \tag{12}$$

$$X^2_{\mathcal{D}^{2+}(X^2),N\setminus\mathcal{D}^1} = X^1_{\mathcal{D}^{2+}(X^2),N\setminus\mathcal{D}^1}. \tag{13}$$

Because X^1 satisfies LP and P and because X^2 satisfies P, by definition of $\mathcal{D}^1$, Equation (12) implies:

$$\forall i \in \mathcal{D}^1 \setminus \mathcal{D}^2, \forall j \in \mathcal{N} \setminus \mathcal{D}^1, L_{ij} = X^2_{ij} = X^1_{ij} = 0. \tag{14}$$

Also, by definition of $\mathcal{D}^{2+}(X^2)$, Equation (13) implies:

$$\forall i \in \mathcal{D}^{2+}(X^2), \forall j \in \mathcal{N} \setminus \mathcal{D}^1, X^2_{ij} = X^1_{ij} = 0. \tag{15}$$

Moreover, since X^2 satisfies LL, Equation (11) implies:

$$X^2_{N\setminus\mathcal{D}^{2=}(X^2),\mathcal{D}^{2=}(X^2)} + e_{\mathcal{D}^{2=}(X^2)} - X^2_{\mathcal{D}^{2=}(X^2),N\setminus\mathcal{D}^{2=}(X^2)} = 0,$$

$$X^2_{N\setminus\mathcal{D}^{2+}(X^2),\mathcal{D}^{2+}(X^2)} + e_{\mathcal{D}^{2+}(X^2)} - X^2_{\mathcal{D}^{2+}(X^2),N\setminus\mathcal{D}^{2+}(X^2)} = 0,$$

$$X^2_{N\setminus(\mathcal{D}_1\setminus\mathcal{D}_2),\mathcal{D}_1\setminus\mathcal{D}_2} + e_{\mathcal{D}_1\setminus\mathcal{D}_2} - X^2_{\mathcal{D}_1\setminus\mathcal{D}_2,N\setminus(\mathcal{D}_1\setminus\mathcal{D}_2)} = 0.$$

4) We showed above that:

$$X^2_{N\setminus\mathcal{D}^{2=}(X^2),\mathcal{D}^{2=}(X^2)} + e_{\mathcal{D}^{2=}(X^2)} - X^2_{\mathcal{D}^{2=}(X^2),N\setminus\mathcal{D}^{2=}(X^2)} = 0,$$

and by X^1 satisfying AP, we have:

$$X^1_{N\setminus\mathcal{D}^{2=}(X^2),\mathcal{D}^{2=}(X^2)} + e_{\mathcal{D}^{2=}(X^2)} - X^1_{\mathcal{D}^{2=}(X^2),N\setminus\mathcal{D}^{2=}(X^2)} = 0.$$

The same reasoning as above shows that:

$$\forall i \in \mathcal{D}^1 \setminus \mathcal{D}^2, \forall j \in \mathcal{D}^{2=}(X^2), L_{ij} = X^1_{ij} = X^2_{ij} = 0, \tag{16}$$

and

$$\forall i \in \mathcal{D}^{2+}(X^2), \forall j \in \mathcal{D}^{2=}(X^2), X^2_{ij} = X^1_{ij} = 0. \tag{17}$$

5) Let us consider $\mathcal{D}^{2+}(X^2) \cup (\mathcal{D}^1 \setminus \mathcal{D}^2)$. Because P^1 satisfies AP,

$$L_{N\setminus\mathcal{D}^1,\mathcal{D}^{2+}(X^2)\cup(\mathcal{D}^1\setminus\mathcal{D}^2)} + X^1_{\mathcal{D}^{2-}(X^2),\mathcal{D}^{2+}(X^2)\cup(\mathcal{D}^1\setminus\mathcal{D}^2)} +$$
$$X^1_{\mathcal{D}^{2=}(X^2),\mathcal{D}^{2+}(X^2)\cup(\mathcal{D}^1\setminus\mathcal{D}^2)} - X^1_{\mathcal{D}^{2+}(X^2)\cup(\mathcal{D}^1\setminus\mathcal{D}^2),N\setminus(\mathcal{D}^{2+}(X^2)\cup(\mathcal{D}^1\setminus\mathcal{D}^2))} + e_{\mathcal{D}^{2+}(X^2)\cup(\mathcal{D}^1\setminus\mathcal{D}^2)} = 0.$$

After simplification using Equations (9), (10), (14), (15), (16), (17):

$$L_{N\setminus\mathcal{D}^1,\mathcal{D}^{2+}(X^2)\cup(\mathcal{D}^1\setminus\mathcal{D}^2)} + X^1_{\mathcal{D}^{2=}(X^2),\mathcal{D}^{2+}(X^2)\cup(\mathcal{D}^1\setminus\mathcal{D}^2)} + e_{\mathcal{D}^{2+}(X^2)\cup(\mathcal{D}^1\setminus\mathcal{D}^2)} = 0.$$

This implies:

$$e_{\mathcal{D}^{2+}(X^2)\cup(\mathcal{D}^1\setminus\mathcal{D}^2)} = 0. \tag{18}$$

and

$$L_{N\setminus\mathcal{D}^1,\mathcal{D}^{2+}(X^2)\cup(\mathcal{D}^1\setminus\mathcal{D}^2)} + X^1_{\mathcal{D}^{2=}(X^2),\mathcal{D}^{2+}(X^2)\cup(\mathcal{D}^1\setminus\mathcal{D}^2)} = 0. \tag{19}$$

The same reasoning with X^2 gives:

$$L_{N\setminus\mathcal{D}^1,\mathcal{D}^{2+}(X^2)\cup(\mathcal{D}^1\setminus\mathcal{D}^2)} + X^2_{\mathcal{D}^{2=}(X^2),\mathcal{D}^{2+}(X^2)\cup(\mathcal{D}^1\setminus\mathcal{D}^2)} = 0. \tag{20}$$

Equations (19) and (20) imply

$$\forall i \in \mathcal{N}\setminus\mathcal{D}^1, \forall j \in \mathcal{D}^{2+}(X^2)\cup(\mathcal{D}^1\setminus\mathcal{D}^2), L_{ij} = X^1_{ij} = X^2_{ij} = 0, \tag{21}$$

an

$$\forall i \in \mathcal{D}^{2=}(X^2), \forall j \in \mathcal{D}^{2+}(X^2)\cup(\mathcal{D}^1\setminus\mathcal{D}^2), X^1_{ij} = X^2_{ij} = 0. \tag{22}$$

Since X^1 and X^2 satisfy LL,

$$\begin{array}{l}\forall i \in \mathcal{D}^{2+}(X^2)\cup(\mathcal{D}^1\setminus\mathcal{D}^2),\\ X^1_{\mathcal{D}^{2+}(X^2)\cup(\mathcal{D}^1\setminus\mathcal{D}^2),i} - X^1_{i,\mathcal{D}^{2+}(X^2)\cup(\mathcal{D}^1\setminus\mathcal{D}^2)} =\\ X^2_{\mathcal{D}^{2+}(X^2)\cup(\mathcal{D}^1\setminus\mathcal{D}^2),i} - X^2_{i,\mathcal{D}^{2+}(X^2)\cup(\mathcal{D}^1\setminus\mathcal{D}^2)} = 0.\end{array} \tag{23}$$

6) From Equation (6), $\mathcal{D}^{2+}(X^2) \neq \emptyset$ and by definition of $\mathcal{D}^{2+}(X^2)$, $\forall k \in \mathcal{D}^{2+}(X^2)$, $X^2_{k,\mathcal{N}} > X^1_{k,\mathcal{N}} \geq 0$. Moreover, since X^2 satisfies P, $\forall k \in \mathcal{D}^{2+}(X^2)$, $L_{k,\mathcal{N}} > 0$. We can then consider $k_{max} \in \mathcal{D}^{2+}(X^2)$ such that $\forall k \in \mathcal{D}^{2+}(X^2)$, $\dfrac{X^2_{k_{max},N}}{L_{k_{max},N}} \geq \dfrac{X^2_{k,\mathcal{N}}}{L_{k,\mathcal{N}}}$.

Assume that $\dfrac{X^2_{k_{max},N}}{L_{k_{max},N}} \leq 1$. This implies that X^2 satisfies LP, contradicting the fact that $\mathcal{D}^1$ is a minimal {LP,P,LL}-solution. Then, consider $\dfrac{X^2_{k_{max},N}}{L_{k_{max},N}} > 1$.

Now, let us define $X' = (X'_{ij})_{i,j\in\mathcal{N}} \in \mathcal{L}$ as:

$$\forall i,j \in \mathcal{N}, X'_{ij} = \begin{cases} X^2_{ij}\dfrac{L_{k_{max},N}}{X^2_{k_{max},N}} & \text{, if } i \in \mathcal{D}^{2+}(X^2)\cup(\mathcal{D}^1\setminus\mathcal{D}^2),\\ X^2_{ij} & \text{, otherwise.}\end{cases}$$

It is straightforward to check that X' satisfies P.

Let us show that X' satisfies LL. a) Let $i \in \mathcal{N}\setminus(\mathcal{D}^{2+}(X^2)\cup(\mathcal{D}^1\setminus\mathcal{D}^2))$. Let us define $\pi(i) = X'_{\mathcal{N},i} + e_i - X'_{i,\mathcal{N}}$. By definition of X', $\pi(i) = X'_{\mathcal{N},i} + e_i - X^2_{i,\mathcal{N}}$. From Equations, (10), (14), (15), (16), and (17), $\pi(i) = X^2_{\mathcal{N},i} + e_i - X^2_{i,\mathcal{N}}$. LL is then satisfied by i for X' as it is for X^2 by assumption. b) Let $i \in \mathcal{D}^{2+}(X^2)\cup(\mathcal{D}^1\setminus\mathcal{D}^2)$. Let us define $\pi(i) = X'_{\mathcal{N},i} + e_i - X'_{i,\mathcal{N}}$. After simplification using Equations (9), (10), (14), (15), (16), (17), (18), (21), and (22), $\pi(i) = X'_{\mathcal{D}^{2+}(X^2)\cup(\mathcal{D}^1\setminus\mathcal{D}^2),i} - X'_{i,\mathcal{D}^{2+}(X^2)\cup(\mathcal{D}^1\setminus\mathcal{D}^2)}$. By definition of X': $\pi(i) = \frac{L_{k_{max},N}}{X^2_{k_{max},N}} X^2_{\mathcal{N},i} - X^2_{i,\mathcal{N}}$. LL is then satisfied by i for X' as it is for X^2 by assumption.

Let us show that X' satisfies LP. a) The proof is straightforward for all $i \in \mathcal{N} \setminus (\mathcal{D}^{2+}(X^2) \cup (\mathcal{D}^1 \setminus \mathcal{D}^2))$. b) Let $i \in \mathcal{D}^{2+}(X^2)$. $X'_{i,\mathcal{N}} = X^2_{i,\mathcal{N}} \frac{L_{k_{max},N}}{X^2_{k_{max},N}}$. Then, by definition of k_{max}, $X'_{i,\mathcal{N}} \leq X^2_{i,\mathcal{N}} \frac{L_{i,\mathcal{N}}}{X^2_{i,\mathcal{N}}} = L_{i,\mathcal{N}}$. LP is then satisfied by i. c) Let $i \in \mathcal{D}^1 \setminus \mathcal{D}^2$. $X'_{i,\mathcal{N}} = X^2_{i,\mathcal{N}} \frac{L_{k_{max},N}}{X^2_{k_{max},N}} = L_{i,\mathcal{N}} \frac{L_{k_{max},N}}{X^2_{k_{max},N}}$. Since, $\frac{L_{k_{max},N}}{X^2_{k_{max},N}} < 1$, $X'_{i,\mathcal{N}} < L_{i,\mathcal{N}}$. LP is then satisfied by i.

It is straightforward to check that, since $(\mathcal{N}\setminus\mathcal{D}^1)\cap(\mathcal{D}^{2+}(P^2)\cup(\mathcal{D}^1\setminus\mathcal{D}^2)) = \emptyset$, $\mathcal{D}(L, X') \subseteq \mathcal{D}(L, X^1)$. It is also straightforward to check that $k_{max} \notin \mathcal{D}(L, X')$, whereas by definition, $k_{max} \in \mathcal{D}^{2+}(X^2) \subseteq \mathcal{D}(L, X^1)$. Hence, $\mathcal{D}(L, X') \subsetneq \mathcal{D}(L, X^1)$, which contradicts the fact that $\mathcal{D}^1$ is a minimal {LP,P,LL}-solution. This completes the proof.

E Proof of Proposition 6

Let $\mathcal{N} = \{1, 2, 3, 4\}$, $e = (1, 2, 3, 4)$, and $L = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{pmatrix}$.

We have $\mathcal{D}_0(e, L) = \{1\}$.

Let us define the CPM $X_1 = \begin{pmatrix} 0 & 6/5 & 12/5 & 6/5 \\ 1 & 0 & 1 & 2 \\ 14/5 & 14/5 & 0 & 14/5 \\ 0 & 0 & 2 & 0 \end{pmatrix}$. X_1 satisfies LL, P, and I and $\mathcal{D}(L, X_1) = \{1, 3\}$ is a minimal {LL,P,I}-solution to (e, L).

Let us define the CPM $X_2 = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 8/3 & 8/3 & 0 & 8/3 \\ 0 & 0 & 2 & 0 \end{pmatrix}$. X_2 satisfies LL and P and $\mathcal{D}(L, X_2) = \{3\}$ is a minimal {LL,P}-solution to (e, L).

Let us define the CPM $X_3 = \begin{pmatrix} 0 & 11/15 & 22/15 & 11/15 \\ 14/15 & 0 & 14/15 & 28/15 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{pmatrix}$. X_3 satisfies LL, P, and LP and $\mathcal{D}(L, X_3) = \{1, 2\}$ is a minimal {LL,P,LP}-solution to (e, L).

$\mathcal{D}(L, X_1)$ is then a minimal {LL,P,I}-solution to (e, L) but is not a minimal {LL,P,LP}-solution to (e, L), proving Proposition 6-1a. $\mathcal{D}(L, X_1)$ is a minimal {LL,P,I}-solution to (e, L) but is not a minimal {LL,P}-solution to (e, L) proving Proposition 6-1b. $\mathcal{D}(L, X_2)$ is a minimal {LL,P}-solution to (e, L) but is not a minimal {LL,P,I}-solution to (e, L), proving Proposition 6-2.