Perron-Frobenius theory recovers more than you might think: The example of limited participation

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Abstract

In their seminal article, Hansen and Scheinkman (2009) proved that Perron-Frobenius theory helps to recover a probability measure that can be used to price long-term claims. In this paper, we show that the recovered probability also contains information about market structure. More precisely, we provide an example in which Perron-Frobenius theory can be used to measure the degree of limited market participation.

Keywords: Perron-Frobenius, Arrow-Debreu securities, limited participation.

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1 Introduction

Asset prices contain information about both stochastic discount factors and transition probabilities. Extracting this information using Perron-Frobenius (PF, henceforth) theory was pioneered by Backus, Gregory, and Zin (1989). More precisely, Hansen and Scheinkman (2009) proved that applying PF theory to Arrow-Debreu (AD, henceforth) security prices makes it possible to recover a probability measure, which provides useful insights into the pricing of long-term claims. As shown in Ross (2015), recovered probability is equal to actual probability under certain specific conditions. Borovička, Hansen, and Scheinkman (2016) have generalized Ross's result and have proven that recovered probability differs from actual probability by a martingale component, which is trivial only under Ross's conditions. Interestingly, most asset pricing models feature a non-trivial martingale component.

In this paper, we provide an example to show that PF theory can be used to recover information not only about long-term pricing but also about market structure. Our example features an economy populated by two heterogeneous agents trading AD securities. Agent heterogeneity generates an endogenous market segmentation. The same agent does not trade all securities in all states of the world and the market arrangement is not the same for all maturities. Applying PF theory in this context delivers two main results. First, the long-term return recovered by PF theory differs from the actual return due to limited market segmentation. The largest eigenvalue of the matrix of AD prices now reflects a long-term discount rate that is distorted by limited participation and agent heterogeneity. Second, the recovered and actual long-term one-period expected holding returns also differ from one another. However, agent's heterogeneity can be recovered from these differences in returns. Consequently, PF theory can help us to quantify the severity of limited market participation. To the best of our knowledge, this is the first paper showing the link between PF theory and financial market structure.

2 Set-up

We consider an economy populated by two agents, A and B, with two states of the world, 1 and 2. The process determining states of the world follows a first-order Markov chain characterized by transition probabilities π and ν, which are the probabilities of remaining in states 1 and 2, respectively. Agents can trade two AD securities. The price in state $i = 1, 2$ of the AD security paying off in state $j = 1, 2$ of the next period is denoted by q_{ij}. The pricing kernel v_{ij}^k of agent $k = A, B$ in state $i = 1, 2$ for a payoff in state $j = 1, 2$ of the next period is assumed to be equal to:

$$v_{ij}^k = \beta^k \frac{m_i^k}{m_j^k}. \tag{1}$$

We introduce two sources of heterogeneity among agents: (i) the coefficient β^k and (ii) the ratio $\frac{m_1^k}{m_2^k}$ (or equivalently $\frac{m_2^k}{m_1^k}$) for $k = A, B$. These two sources are the minimum components required to generate a nontrivial market structure, which also differs for securities with different maturities. Such a set-up could, for instance, result from a no-trade equilibrium in an economy featuring heterogeneous agents (in terms of β and endowments), credit constraints, and zero net asset supply. See for instance Krusell, Mukoyama, and Smith (2011) or Challe, LeGrand, and Ragot (2013) for such set-ups.

The framework of this example is a slight deviation from (see Borovička, Hansen, and Scheinkman, 2016; Ross, 2015). Note that Ross's recovery result holds here when heterogeneity is absent, i.e., when $\beta^A = \beta^B$ and $\frac{m_1^A}{m_2^A} = \frac{m_1^B}{m_2^B}$.

We introduce the following notation:

$$\tau^m = \frac{\beta^A}{\beta^B} \frac{m_2^B}{m_1^B} \frac{m_1^A}{m_2^A}, \quad \tau^\beta = \frac{\beta^A}{\beta^B}. \tag{2}$$

The two quantities τ^m and τ^β summarize the two heterogeneity dimensions and measure the deviation of the actual economy from the recovery setup, which corresponds to $\tau^\beta = \tau^m = 1$. We also make the following assumption:

Assumption A *We assume that $\tau^\beta \leq 1$ and $\tau^m \geq 1$.*

The first part of Assumption A ($\tau^{\beta} \leq 1$) means that agent B is more patient than agent A. The second part ($\tau^{m} \geq 1$) states that the marginal rate of substitution between state 2 now and state 1 in the next period is larger for agent A than for agent B.

Assumption A enables us to generate endogenous limited participation and a nontrivial market structure. It implies that the AD security paying off in state 2 is always traded by agent B, while the other AD security, paying off in state 1, is traded by agent A in state 2 and by agent B in state 1.We deduce that the matrix $Q = (q_{ij})_{i,j=1,2}$ of AD security prices can be expressed as:

$$Q = \beta^{B} \begin{pmatrix} \pi & (1-\pi)\frac{m_2^B}{m_1^B} \\ (1-\nu)\tau^{m}\frac{m_1^B}{m_2^B} & \nu \end{pmatrix}. \tag{3}$$

3 Using Perron-Frobenius theory to recover limited participation

In contrast with the existing literature, we prove that the long-term return recovered by PF from one-period AD security prices differs from the actual long-term return. Interestingly, PF theory can be used to recover the severity of market segmentation and in particular its determinants, τ_{β} and τ_{m}.

We now consider AD securities paying off in n periods. The price in state i of an AD security paying off in state j in n periods is denoted by $q_{ij}^{(n)}$.

3.1 A preliminary lemma

To avoid discussing too many cases, we make the following assumption.

Assumption B *We assume that agents' pricing kernels are such that:*

$$m_B^1 \geq m_B^2,$$
$$(\tau^m - 1)\,\pi \geq \nu(1 - \tau^\beta),$$
$$(1-\nu)(1-\pi)\,(\tau^m - 1) \geq \nu^2(1-\tau^\beta)\tau^\beta.$$

Assumption B is compatible with Assumption A and holds when the ratio $\frac{m_1^A}{m_2^A}$ is sufficiently large, or when the ratio τ^β is sufficiently close to one. This condition can be interpreted as the fact that the heterogeneity caused by $\frac{m_2^k}{m_1^k}$ is "stronger" than that caused by β^k.

Assumption B looks complicated but is simply meant to generate a non-trivial market structure for AD securities with maturity greater than 2 periods. The next lemma summarizes the market structure implied by Assumption B.

Lemma 1 (Market structure) *If Assumption B holds, for any AD security of maturity $n \geq 2$, agent A is the price-maker in state 2, while agent B is the price-maker in state 1.*

Proof of Lemma 1, as all other proofs, can be found in Appendix. Assumption B implies perfect market segmentation for AD securities with maturity greater than two periods. Only agent A trades in state 2, while only agent B trades in state 1. The market segmentation for securities with maturity greater than two periods differs from the market arrangement observed for one-period AD securities. This difference in market segmentation for AD securities with different maturities is key to explaining why limited participation affects the long-term rate recovered using PF theory.

3.2 First dimension: Recovering τ^β

We now state our first result relating to the impact of limited market participation on long-term asset returns.

Proposition 1 (Limited market participation) *The actual long-term return r_∞^a differs from the return recovered from one-period AD prices using PF theory, r_∞^{PF}.*

For any τ^m, the difference between the long-term returns, denoted by $\delta r_\infty = r_\infty^{PF} - r_\infty^a$, is always nonpositive and increases with τ^β. Furthermore:

- *when $\tau^\beta \to 1$, $\delta r_\infty \to 0$;*
- *when $\tau^\beta \to 0$, $\delta r_\infty \to \ln\left(\frac{\pi+\left(\pi^2+4(1-\pi)(1-\nu)\tau^m\right)^{\frac{1}{2}}}{\pi+\nu+((\pi-\nu)^2+4(1-\pi)(1-\nu)\tau^m)^{\frac{1}{2}}}\right) < 0$.*

Proposition 1 shows that applying PF theory to one-period AD security prices fails to yield an actual long-term return. More precisely, the proposition guarantees the existence of a one-to-one relationship between dimension τ^β of limited market participation and the long-term return gap δr_∞. The difference between actual and recovered returns thus indirectly recovers dimension τ^β of limited market participation.

The intuition for the Proposition 1 result can be explained by PF theory. The long-term rate is usually characterized by the largest eigenvalue of the matrix of one-period AD prices (here, Q); however, due to the market structure described in Lemma 1, this does not hold in our economy. The long-term rate is determined by the largest eigenvalue of another matrix, which differs from Q.

3.3 Second dimension: Recovering τ^m

We now take advantage of the long-term one-period expected holding returns (EHR, henceforth) to recover the heterogeneity in τ^m. The price in state i of the n-period zero-coupon bond is $p_i^{(n)} = q_{i1}^{(n)} + q_{i2}^{(n)}$. The EHR $r_{i,(n)}^{1,\Pi}$ is defined as the expected return for purchasing a n-period bond in state i and reselling it in one period as a $(n-1)$-period bond. More formally, we have:

$$r_{i,(n)}^{1,\Pi} = \frac{\Pi_{i1} p_1^{(n-1)} + \Pi_{i2} p_2^{(n-1)}}{p_i^{(n)}}.$$

The average EHR $r_{(n)}^{1,\Pi}$ is equal to the unconditional average of the state-dependent

EHR:

$$r_{(n)}^{1,\Pi} = \frac{1-\Pi_{22}}{2-\Pi_{11}-\Pi_{22}} r_{1,(n)}^{1,\Pi} + \frac{1-\Pi_{11}}{2-\Pi_{11}-\Pi_{22}} r_{2,(n)}^{1,\Pi}.$$

We denote by $r_{(n)}^1$ the EHR when the transition probabilities are actual probabilities, while $\widetilde{r}_{(n)}^1$ corresponds to the EHR when the transition probabilities are recovered by PF theory. We also denote by r_∞^1 and $\widetilde{r}_\infty^1$ their respective limits when the maturity n becomes very long and converges to infinity. We now state the following proposition.

Proposition 2 (One-period holding return) *The ratio of EHR,* $\frac{\widetilde{r}_\infty^1}{r_\infty^1}$*, is an increasing function of* τ^m*. In particular,* $\widetilde{r}_\infty^1 \geq r_\infty^1$*.*

Proposition 2 shows that the heterogeneity in τ^m widens the difference between the EHRs under the recovered and actual probability measures. Moreover, the recovered EHR is always greater than the actual EHR: the recovered probability puts greater weight on larger returns than is actually the case. Finally, Proposition 2 mirrors Proposition 1 and shows that EHRs can be used to recover dimension τ^m of agents' heterogeneity, while keeping the other dimension τ^β unchanged.

3.4 Recovering the determinants of market segmentation

Propositions 1 and 2 have established partial invertibility results stating that the observation of either long-term returns or long-term one-period holding returns is sufficient to recover one of the heterogeneity determinants τ^β or τ^m. We now state a global invertibility result showing that observing both long-term returns and long-term EHR enables us to jointly recover both determinants of market segmentation.

Proposition 3 (Recovering (τ^β, τ^m)) *For a given observation of long-term return difference* $r_\infty^{PF} - r_\infty^a$ *and long-term EHR ratio* $\frac{\widetilde{r}_\infty^1}{r_\infty^1}$*, there is a unique pair* (τ^β, τ^m)*.*

This result shows that the application of PF theory allows us to completely quantify limited financial participation in our example economy. It generalizes the results of Propositions 1 and 2 to the joint recovery of the determinants of market segmentation.

4 Conclusion

We have shown that applying PF theory in a context of limited market participation provides a distorted version of the long-term return. However, PF theory enables us to recover the determinants of limited market participation and to understand the underlying heterogeneity of market participants. To the best of our knowledge, this paper is the first to illustrate the possible connection between PF and financial market structure. We leave the exploration of the general theory of PF with limited participation for future research.

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Appendix

A Proof of Lemma 1

We introduce the two following notations:

$$Q_A = \beta^B \begin{pmatrix} \pi & (1-\pi)\frac{m_2^B}{m_1^B} \\ (1-\nu)\frac{m_1^B}{m_2^B}\tau^m & \nu\tau^\beta \end{pmatrix} \text{ and } Q_B = \beta^B \begin{pmatrix} \pi & (1-\pi)\frac{m_2^B}{m_1^B} \\ (1-\nu)\frac{m_1^B}{m_2^B} & \nu \end{pmatrix}. \quad (4)$$

The matrix Q_A collects (possibly out-of-equilibrium) AD prices when B is price-maker in state 1 and A price maker in state 2. Similarly, the matrix Q_B is the matrix of AD prices when B is price-maker in both states. We prove that $(Q_A - Q_B)Q_A^n Q$ has only positive elements for all n. Diagonalizing Q_A yields $Q_A = E_A D_A E_A^{-1}$ with:

$$\begin{aligned} D_A &= \operatorname{diag}(K_1(\tau^\beta, \tau^m), K_2(\tau^\beta, \tau^m)), \\ E_A &= \beta^B \begin{pmatrix} (1-\pi)\frac{m_1^B}{m_2^B} & (1-\pi)\frac{m_1^B}{m_2^B} \\ \frac{1}{2}\left(\nu\tau^\beta - \pi + \sqrt{\Delta(\tau^\beta, \tau^m)}\right) & \frac{1}{2}\left(\nu\tau^\beta - \pi - \sqrt{\Delta(\tau^\beta, \tau^m)}\right) \end{pmatrix}, \\ \Delta(\tau^\beta, \tau^m) &= (\pi - \nu\tau^\beta)^2 + 4(1-\pi)(1-\nu)\tau^m, \\ K_i(\tau^\beta, \tau^m) &= \frac{\beta^B}{2}\left(\pi + \nu\tau^\beta + (-1)^{i-1}\sqrt{\Delta(\tau^\beta, \tau^m)}\right), \ i = 1, 2, \end{aligned}$$

Dropping the dependence in (τ^β, τ^m), we have $(Q_A - Q_B)Q_A^n Q = \begin{pmatrix} 0 & 0 \\ d_{1,n} & d_{2,n} \end{pmatrix}$, where:

$$\begin{aligned} \frac{\frac{m_1^A}{m_2^A} d_{1,n}}{\beta^A \beta^B (1-\nu)} &= \left((1 - 1/\tau^m)\,\pi - \nu(1-\tau^\beta)\right)(\lambda_{A,1}^{n+1} - \lambda_{A,2}^{n+1}) \qquad (5) \\ &+ \beta^A \left((1-\pi)(1-\nu)\tau^\beta\,(\tau^m - 1) - \nu\pi\,(1 - 1/\tau^m)\right)(\lambda_{A,1}^n - \lambda_{A,2}^n), \\ \frac{d_{2,n}}{(\beta^B)^2} &= \left((1-\nu)(1-\pi)\,(\tau^m - 1) - \nu\nu(1-\tau^\beta)\right)(\lambda_{A,1}^{n+1} - \lambda_{A,2}^{n+1}) \qquad (6) \\ &+ \nu(1-\tau^\beta)\,(\nu + \pi - 1)\,\beta^B(\lambda_{A,1}^n - \lambda_{A,2}^n). \end{aligned}$$

Before going further, let us remark that:

$$\lambda_{A,1}^{n+1} - \lambda_{A,2}^{n+1} \geq \nu\beta^A(\lambda_{A,1}^n - \lambda_{A,2}^n), \tag{7}$$

$$\geq \pi\beta^B(\lambda_{A,1}^n - \lambda_{A,2}^n). \tag{8}$$

Let us prove (7) (similar for (8)). Note that (i) $\lambda_{A,2} < \lambda_{A,1}$ and (ii) $0 \leq \nu\beta^A \leq \lambda_{A,1}$. First, since $\lambda_{A,1} > 0$, (7) is equivalent to $1 - \frac{\nu\beta^A}{\lambda_{A,1}} \geq \frac{\lambda_{A,2}^n}{\lambda_{A,1}^n}(\frac{\lambda_{A,2}}{\lambda_{A,1}} - \frac{\nu\beta^A}{\lambda_{A,1}})$. The result holds when$\lambda_{A,2} \geq 0$ or when $\lambda_{A,2} < 0$ and n is even. Now assume that $n = 2m+1$ and $\lambda_{A,2} < 0$. The sequence $m \mapsto \frac{\lambda_{A,2}^{2m+1}}{\lambda_{A,1}^{2m+1}}(\frac{\lambda_{A,2}}{\lambda_{A,1}} - \frac{\nu\beta^A}{\lambda_{A,1}})$ is positive and decreasing since $\frac{\lambda_{A,2}^2}{\lambda_{A,1}^2} \in [0,1)$. We conclude by proving that (7) holds for $m = 0$. Indeed, it is equivalent to $\lambda_{A,1} + \lambda_{A,2} \geq \nu\beta^A$, which holds.

Now, using (5) and (7), $d_{1,n}$ becomes $\frac{d_{1,n}}{\beta^A\beta^B(1-\nu)(\lambda_{A,1}^n - \lambda_{A,2}^n)\beta^A \frac{m_2^A}{m_1^A}} \geq (1-\pi)(1-\nu)\left(\tau^m - 1\right)/\tau^\beta - \nu^2(1-\tau^\beta) \geq 0$, with Assumption B. Finally, we similarly prove that $d_{2,n} \geq 0$.

B Proof of Proposition 1

The largest eigenvalue of one-period AD securities is $\lambda_Q = \frac{\beta^B}{2}\left(\pi + \nu + \Delta^{1/2}(1,\tau^m)\right)$. However, due to Lemma 1, the long-term return of an AD security depends on $\lambda_{Q_A} = \frac{\beta^B}{2}\left(\pi + \nu\tau^\beta + \Delta^{1/2}\right)$. The actual long-term rate is thus $r_\infty^a = -\log(\lambda_{Q_A})$, while the recovered one is $r_\infty^{PF} = -\log(\lambda_Q)$. The difference is $\delta r_\infty = r_\infty^{PF} - r_\infty^a = \log\left(\frac{\lambda_{Q_A}}{\lambda_Q}\right)$, where:

$$\frac{\lambda_{Q_A}}{\lambda_Q} = \frac{\pi + \nu\tau^\beta + \left((\pi - \nu\tau^\beta)^2 + 4(1-\pi)(1-\nu)\tau^m\right)^{\frac{1}{2}}}{\pi + \nu + \left((\pi - \nu)^2 + 4(1-\pi)(1-\nu)\tau^m\right)^{\frac{1}{2}}}. \tag{9}$$

We denote δr_∞ as $\varphi(\tau^\beta, \tau^m)$, with:

$$\frac{\partial\varphi(\tau^\beta,\tau^m)}{\partial\tau^\beta} = -C \times \left(1 - (\pi - \nu/\tau^\beta)\left((\pi - \nu/\tau^\beta)^2 + 4(1-\pi)(1-\nu)\tau^m\right)^{-\frac{1}{2}}\right), \tag{10}$$

where $C > 0$. We deduce that δr_∞ increases with τ^β and since $\varphi(1,\tau^m) = 1$, $\delta r_\infty \leq 0$.

C Proof of Proposition 2

PF theory for AD prices in matrix Q of (3) implies that the recovered probability $\widetilde{\mathbb{P}}$ is characterized by the transition matrix $\widetilde{P}$:

$$\widetilde{P} = \begin{pmatrix} \widetilde{\pi} & 1-\widetilde{\pi} \\ 1-\widetilde{\nu} & \widetilde{\nu} \end{pmatrix}, \tag{11}$$

$$\text{where: } \widetilde{\pi} = \frac{2\pi}{\pi+\nu+\sqrt{\Delta_1}},\ \widetilde{\nu} = \frac{2\nu}{\pi+\nu+\sqrt{\Delta_1}},\ \text{and } \Delta_1 = \Delta(1,\tau_m). \tag{12}$$

The price $p_i^{(n)}$ in state $i = 1, 2$ of a zero-coupon bond of maturity n periods is:

$$p_i^{(n)} = \lambda_{Q_A}^{n-1}\frac{\beta^B}{\sqrt{\Delta}}\rho_i(1+o_n(1)), \quad i = 1,2, \tag{13}$$

$$\rho_1 = \frac{\pi-\nu\tau^\beta+\sqrt{\Delta}}{2}\pi + (1-\pi)(1-\nu)\tau^m + (1-\pi)\frac{m_2^B}{m_1^B}\frac{\pi+\nu(2-\tau^\beta)+\sqrt{\Delta}}{2}, \tag{14}$$

$$\rho_2 = \frac{-\pi+\nu\tau^\beta+\sqrt{\Delta}}{2}\nu + (1-\pi)(1-\nu)\tau^m + (1-\nu)\tau^m\frac{m_1^B}{2m_2^B}\left(\pi+\nu\tau^\beta+\sqrt{\Delta}\right), \tag{15}$$

where $o_n(1) \to_{n\to\infty} 0$. After some algebra, (14) becomes:

$$\rho_1 = \frac{1}{2}\left(-\pi+\nu\tau^\beta+\sqrt{\Delta_A}\right)\frac{m_1^B}{(1-\pi)m_2^B}\rho_2. \tag{16}$$

We denote $r_{(n)}^1$ the one-period average EHR for a n-period bond under $\mathbb{P}$, while $\widetilde{r}_{(n)}^1$ denotes the same return under $\widetilde{\mathbb{P}}$. Using (13), we obtain:

$$r_{(n)}^1 = \frac{2}{\beta((\pi+\nu)+\sqrt{\Delta})}\left(1+\frac{(1-\pi)(1-\nu)}{2-\pi-\nu}\frac{(\rho_1-\rho_2)^2}{\rho_1\rho_2}\right) + o_n(1), \tag{17}$$

$$\widetilde{r}_{(n)}^1 = \frac{2}{\beta((\pi+\nu)+\sqrt{\Delta})}\left(1+\frac{(1-\widetilde{\pi})(1-\widetilde{\nu})}{2-\widetilde{\pi}-\widetilde{\nu}}\frac{(\rho_1-\rho_2)^2}{\rho_1\rho_2}\right) + o_n(1). \tag{18}$$

We deduce from (17) and (18) $\frac{\tilde{r}^1_{(n)}}{r^1_{(n)}} = 1 + \frac{(1-\tilde{\pi})(1-\tilde{\nu})}{2-\tilde{\pi}-\tilde{\nu}} \frac{(\rho_1-\rho_2)^2}{\rho_1\rho_2} \Big/ 1 + \frac{(1-\pi)(1-\nu)}{2-\pi-\nu} \frac{(\rho_1-\rho_2)^2}{\rho_1\rho_2}$, and:

$$\frac{\tilde{r}^1_{(n)}}{r^1_{(n)}} = \psi(\tau^m) = \frac{1 + g(\tau^m)f(\tau^m)}{1 + f(\tau^m)}, \tag{19}$$

$$\text{where: } g(\tau^m) = \frac{2(2-\pi-\nu)\tau^m}{\sqrt{\Delta(\tau^m)}(\pi+\nu+\sqrt{\Delta(\tau^m)})}, \tag{20}$$

$$f(\tau^m) = \frac{(1-\pi)(1-\nu)}{2-\pi-\nu}\left(\frac{\rho_1(\tau^m)}{\rho_2(\tau^m)} + \frac{\rho_2(\tau^m)}{\rho_1(\tau^m)} - 2\right). \tag{21}$$

Using (19) we have:

$$\psi'(\tau^m) = \frac{g'(\tau^m)f(\tau^m)(1+f(\tau^m)) + f'(\tau^m)(g(\tau^m)-1)}{(1+f(\tau^m))^2}. \tag{22}$$

If g and f are increasing, we have $g(\tau^m) \geq 1$ for any $\tau^m \geq 1$ since $g(1) = 1$. Since $g, f > 0$, (22) implies $\psi'(\tau^m) > 0$ and ψ strictly increasing on $[1, \infty)$. This also implies $\psi(\tau^m) \geq 1$ and $\tilde{r}^1_{(n)} \geq r^1_{(n)}$.

We now show that g is increasing. From (20):

$$-\frac{\partial}{\partial \tau^m}\ln(g(\tau^m)) = \frac{\Delta'(\tau^m)}{2\Delta(\tau^m)} + \frac{\Delta'(\tau^m)}{2\sqrt{\Delta(\tau^m)}(\pi+\nu+\sqrt{\Delta(\tau^m)})} - \frac{1}{\tau^m}.$$

Eq. (12) yields $-\frac{\partial}{\partial \tau^m}\ln(g(\tau^m)) \leq \frac{-(\pi-\nu)^2}{\tau^m\Delta(\tau^m)} < 0$, which implies that g is increasing.

We now consider the case of f. Using (16), we have:

$$f(\tau^m) = \frac{(1-\pi)(1-\nu)}{2-\pi-\nu}\left(f_1(\tau^m) + \frac{1}{f_1(\tau^m)} - 2\right), \tag{23}$$

$$\text{where: } f_1(\tau^m) = \frac{1}{2}\left(-\pi + \nu/\tau^\beta + \sqrt{\Delta_A}\right)\frac{m_1^B}{(1-\pi)m_2^B}. \tag{24}$$

Since $f > 0$, we deduce from (23) that the sign of f' is given by the sign of $f_1'(\tau^m)(f_1(\tau^m)^2 - 1)$. From (12) and (24), we deduce that $f_1'(\tau^m) > 0$. This implies that $f_1(\tau^m) \geq 1$ because of Assumption B. We deduce that f is increasing, which concludes the proof.

D Proof of Proposition 3

We assume that we observe the long-term return difference δr_∞ and of long-term EHR ratio $\frac{\tilde{r}^1_{(n)}}{r^1_{(n)}}$. From Propositions 1 and 2, we know that there exist $a, b > 0$ such that $\delta r_\infty = e^{-b}$ and $\frac{\tilde{r}^1_{(n)}}{r^1_{(n)}} = e^b$. Using (9) and (19), a pair (τ^β, τ^m) matches observed returns if:

$$\begin{cases} \varphi(\tau^\beta, \tau^m) & = e^{-b}, \\ \psi(\tau^\beta, \tau^m) & = e^a. \end{cases} \tag{25}$$

We now prove that at most one pair (τ^β, τ^m) solves (25). Substituting expressions of φ and ψ, (25) is equivalent to:

$$\begin{cases} \pi + \nu\tau^\beta + \Delta(\tau^\beta, \tau^m) & = e^{-b}\left(\pi + \nu + \Delta(1, \tau^m)\right), \\ \frac{1+g(\tau^m)f(\tau^\beta,\tau^m)}{1+f(\tau^\beta,\tau^m)} & = e^a. \end{cases} \tag{26}$$

where f and f_1 in (23) and (24) are generalized to two variables. Let us define $\hat{f}_1(\tau^m) = \frac{1}{2}\left(e^{-b}\left(\pi + \nu + \Delta(1, \tau^m)\right) - 2\pi\right)\frac{m^B_1}{(1-\pi)m^B_2}$ and $\hat{f}(\tau^m) = \frac{(1-\pi)(1-\nu)}{2-\pi-\nu}\left(\hat{f}_1(\tau^m) + \frac{1}{\hat{f}_1(\tau^m)} - 2\right)$. By construction, $\hat{f}_1$ and $\hat{f}$ are increasing functions. We deduce that (25) implies:

$$\frac{1 + g(\tau^m)\hat{f}(\tau^m)}{1 + \hat{f}(\tau^m)} = e^a, \tag{27}$$

where the left hand-side is increasing (same as for ψ defined in (19)). The solution τ^m of (27) is thus unique (if exists). We also know from Proposition 1 that the solution in τ^β of $\varphi(\tau^\beta, \tau^m) = e^{-b}$ for any τ^m is unique. Thus, the solution (τ^β, τ^m) to (25) is unique.