Ambiguity or Endogenous Discounting?

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Abstract

The paper considers a class of preferences over stochastic consumption streams which satisfy an extension of Koopmans' classical stationarity axiom. A notion of ambiguity aversion and two representation theorems for this class of preferences are provided. As part of the analysis, the paper identifies intertemporal behavior that can help discriminate between dynamic models of ambiguity aversion and expected utility models with endogenous discounting. While it is known that under both types of models, individuals are averse to positive autocorrelation in their consumption profile, this paper identifies a particular form of autocorrelation that is disliked by ambiguity averse agents only.

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1 Introduction

Consider the following two generalizations of the standard model of intertemporal choice:

$$V(c_0, c_1, ...) = \mathbb{E}_p[u(c_0) + b(c_0)u(c_1) + b(c_0)b(c_1)u(c_2)...] \tag{1.1}$$

$$V(c_0, c_1, ...) = \min_{p\in\mathcal{P}} \mathbb{E}_p[\textstyle\sum_t \beta^t u(c_t)] \tag{1.2}$$

where $(c_0, c_1, ...)$ is a stochastic consumption stream. Both models have featured prominently in applied work. The first one relaxes time additivity by introducing a specific form of intertemporal complementarity: the rate of time preference depends on the consumption path. It is common to say that discounting is endogenous. A special case of this model was first introduced by Uzawa [32] in a setting without uncertainty. The model in (1.1), and the extension to a stochastic framework in particular, is due to Epstein [9]. The model has found applications in many settings, most notably in the study of small open economies. See Epstein and Hynes [11] for a survey.

In (1.2), the ranking of deterministic consumption streams is represented by a time additive utility function and discounting takes the familiar geometric form. What is different is the presence of ambiguity: the individual cannot quantify the relevant uncertainty in terms of a single prior belief; instead, he contemplates a set $\mathcal{P}$ of beliefs and evaluates each consumption profile according to a worst case scenario. As is known from Ellsberg [8] and Gilboa and Schmeidler [18], the presence of ambiguity leads to non-separabilities across the states of the world. A recent survey of the theoretical, as well as the applied, literature on ambiguity aversion is provided in Epstein and Schneider [12].

The starting point of this paper is that the two models, despite their obvious differences share a number of important predictions concerning *intertemporal* behavior. Most notably, both models exhibit what Epstein [9] calls *correlation aversion* and what Kochov [19] calls *intertemporal hedging.* The intuitive meaning of such behavior is that the individual seeks to take different, negatively correlated bets in different time periods. Kochov [19] shows that intertemporal hedging can be viewed as the dynamic manifestation of ambiguity aversion. A limitation of that paper is that its conclusions depend critically on the assumption that the ranking of deterministic consumption streams be time separable, as in (1.2). In particular, it is known from Epstein [9] that the model in (1.1) generates similar type of hedging behavior whenever the function $b(\cdot)$ is decreasing in the level of consumption. The latter restriction, known as **increasing marginal impatience**, is especially common in applications where it is used to guarantee the uniqueness and stability of steady states. The question arises: if we observe intertemporal hedging, is the behavior due to ambiguity aversion, as in (1.2), or an endogenous discounting, as in (1.1)?

We answer this question by identifying a specific type of intertemporal hedging that is exhibited by ambiguity averse agents only. The answer is based on a distinction between two forms of uncertainty that can arise in a dynamic setting. As an illustration, suppose that the consumption good is one dimensional. Then, in a static environment in which consumption takes place at a single point in time, uncertainty can only affect the level of consumption, that is, *how much* is ultimately consumed. In a dynamic environment, one can imagine a different type of uncertainty. Think of an individual expecting a tax refund. The individual knows the size of the refund and plans to consume the refund as soon as it arrives. The uncertainty he faces is *when* the tax refund will arrive. We show that "an intertemporal hedge" against this specific type of uncertainty - about the timing of consumption - brings out a sharp difference between the models in (1.1) and (1.2). In particular, note that uncertainty about the timing of a tax refund arises year after year. Moreover, the exact timing within year t and the exact timing within year $t+1$ may be positively or negatively correlated. Suppose that the individual prefers negative to positive correlation. We show that the model in (1.1) cannot explain such behavior. By comparison, such behavior is consistent with models of ambiguity aversion, such as the maxmin model in (1.2).

The preceding analysis is complemented by two representation theorems. It is known from Kochov [19] and Epstein [9] that the models in (1.1) and (1.2) are both **path stationary**. As we explain in Section 7.3, path stationarity provides one possible way by which to extend Koopmans' [21] classical notion of stationarity from a setting without uncertainty, as in Koopmans [21], to a stochastic setting. Theorem 1 in this paper provides a representation for all path stationary preferences.[1]

To understand path stationarity, consider an event which resolves in period t and note that the event may affect consumption in any two subsequent periods: $t+k$ or $t+k'$. Path stationarity requires that the individual exhibits the same attitude toward uncertainty no matter whether the event affects consumption in period $t+k$ or $t+k'$. Thus, attitudes toward uncertainty do not depend on the date on which

[1] Epstein [9] and Kochov [19] use the term stationary to refer to what we call path stationarity. We adopt a different term so as to distinguish two ways by which one can extend Koopmans' notion of stationarity to a stochastic setting. In particular, we use path stationarity for the extension we study presently and reserve the term stationarity for another extension discussed in Section 7.3.

consumption takes place.

Path stationarity proves to be a remarkably powerful restriction on behavior, delivering a representation with two specific features. First, the discounted lifetime utility of a non-stochastic consumption stream $(c_0, c_1, \ldots)$ is computed in the manner of (1.1):

$$U(c_0, c_1, \ldots) = u(c_0) + b(c_0)u(c_1) + b(c_0)b(c_1)u(c_2) + \ldots \tag{1.3}$$

When the consumption stream is stochastic, lifetime utility becomes a random variable and the individual needs to compute expectations. Let Ω be a state space and $\xi : \Omega \to \mathbb{R}$ a random variable. Let I be a certainty equivalent, that is, a function that maps each random variable ξ into a real number interpreted as the expected value of ξ. The second implication of path stationarity is that expectations are computed according to a certainty equivalent I that is both translation invariant and positively homogeneous:

$$I(\xi + k) = I(\xi) + k \qquad \text{and} \qquad I(\alpha\xi) = \alpha I(\xi) \tag{1.4}$$

for all $\xi : \Omega \to \mathbb{R}, k \in \mathbb{R}$, and $\alpha \in \mathbb{R}_+$. Such certainty equivalents I are known as *invariant biseparable* and have played a prominent role in the ambiguity literature. See Ghirardato et al. [17] and the references therein. The standard expectation operator in (1.1) and the more general "maxmin certainty equivalent" in (1.2) are both invariant biseparable. The novelty in this paper is to characterize such certainty equivalents by focusing on intertemporal behavior and path stationarity in particular. This is in contrast to the preceding literature which has focused predominantly on static choice.

Path stationarity does not restrict whether the individual is ambiguity neutral, ambiguity averse, or ambiguity loving. One can capture each possibility by choosing the certainty equivalent I appropriately.[2] It is thus natural to ask if the type of intertemporal hedging identified in this paper is sufficiently strong to impose further structure on the certainty equivalent I. Our second representation theorem shows that this is indeed the case. Such hedging is equivalent to having a maxmin certainty equivalent.

$$I(\xi) = \min\nolimits_{p \in P} \mathbb{E}_p \xi, \tag{1.5}$$

[2]For example, if the min in (1.2) is replaced with a max, one obtains a path stationary, ambiguity loving preference relation. Note also that both specifications of the certainty equivalent satisfy the properties in (1.4).

where P is a set of probability measures on the state space Ω. Thus, the notion of intertemporal hedging we propose is not only indicative of ambiguity aversion, it characterizes fully the use of a maxmin certainty equivalent.

2 Domain

Time is discrete and varies over an infinite horizon: $t \in \{0, 1, 2, ...\} =: T$. The available information is described by a filtered space $(\Omega, \{\mathcal{F}_t\}_t)$ where Ω is an arbitrary set of states of the world and $\{\mathcal{F}_t\}_t =: \mathcal{F}$ is an increasing sequence of algebras such that $\mathcal{F}_0 = \{\Omega, \varnothing\}$. Let X be a compact, connected, and separable topological space, interpreted as the set of consumption outcomes.[3] Let h be an X-valued, $\mathcal{F}$-adapted process, that is, a sequence $(h_t)_{t \in T}$ such that $h_t : \Omega \to X$ is $\mathcal{F}_t$-measurable for every $t \in T$. Think of h as a stochastic consumption stream. Following Savage [30], we may also refer to a process h as an **act**. Given an algebra $\mathcal{F}' \subset \cup_t \mathcal{F}_t$, an act h is $\mathcal{F}'$-**adapted** if h_t is $\mathcal{F}'$-measurable for every $t \in T$. An act h is **finite** if there is a finite algebra $\mathcal{F}' \subset \cup_t \mathcal{F}_t$ such that h is $\mathcal{F}'$-adapted. To avoid technical complications, we take the choice domain to be the space $\mathcal{H}$ of all finite acts. An act $h \in \mathcal{H}$ is **deterministic** if, for every $t \in T$, $h_t : \Omega \to X$ is a constant function, that is, if the outcomes of h do not depend on the state of the world. We use $d, d' \in \mathcal{H}$ to denote such acts. As is common in the literature, such acts are identified with elements of X^∞.

Let B^0 be the space of all simple, real valued, $\cup_t \mathcal{F}_t$-measurable functions on Ω. Endow B^0 with the sup norm. Given a set $C \subset \mathbb{R}$, let B^0_C denote the set of all C-valued functions in B^0. To highlight the fact that Ω is a state space, we refer to the functions $\xi \in B^0$ as random variables. We abuse notation and use k to denote both a real number and the function in B^0 that is identically equal to $k \in \mathbb{R}$. With this in mind, a function $I : B^0 \to \mathbb{R}$ is **translation invariant** if $I(\xi + k) = I(\xi) + k$ for all $\xi \in B^0, k \in \mathbb{R}$. It is **normalized** if $I(k) = k$ for all $k \in \mathbb{R}$. Given $\alpha \in \mathbb{R}$, the function I is α-**homogeneous** if $I(\alpha\xi) = \alpha I(\xi)$ for all $\xi \in B^0$. If I is α-homogeneous for all $\alpha \in \mathbb{R}_{++}$, then I is **positively homogeneous**. If we endow B^0 with the usual pointwise order, a function $I : B^0 \to \mathbb{R}$ is **increasing** if for all $\xi, \xi' \in B^0$, $\xi \geq \xi'$ implies that $I(\xi) \geq I(\xi')$. A normalized, increasing and norm continuous function $I : B^0 \to \mathbb{R}$ is called a **certainty equivalent**. In later sections, certainty equivalents are used

[3] Compactness is not essential. See Kochov [19, p.240] for details. Connectedness is needed for our representation theorems but not for the formulation of the axioms. See Section 7.1 for further discussion of this point.

to specify the individual's "beliefs". Thus, $I(\xi)$ is interpreted as the expected value assigned by the individual to the random variable $\xi \in B^0$. Finally, let $\Delta(\Omega)$ be the space of all finitely additive probability measures p on the measurable space $(\Omega, \cup_t \mathcal{F}_t)$. The space $\Delta(\Omega)$ is endowed with the weak* topology, that is, the coarsest topology such that for every $\xi \in B^0$, the linear function $p \mapsto \mathbb{E}_p \xi$ from $\Delta(\Omega)$ into $\mathbb{R}$ is continuous.

A **preference relation** $\succeq$ on a set Y is a complete and transitive binary relation such that $y \succ y'$ for some $y, y' \in Y$. If Y is a topological space, then $\succeq$ is **continuous** if the upper and lower contour sets, $\{y' \in Y : y' \succeq y\}$ and $\{y' \in Y : y \succeq y'\}$, are closed for every $y \in Y$.

3 Path Stationary Preferences

3.1 Axioms

The behavioral primitive in this paper is a preference relation $\succeq$ on the space $\mathcal{H}$ of finite acts. The first restriction we impose on $\succeq$ is a form of continuity familiar from Ghirardato and Marinacci [15]. To state the axiom, endow $\mathcal{H}$ with the product topology.

Finite Continuity (FC): For every finite algebra $\mathcal{F}' \subset \cup_t \mathcal{F}_t$, the restriction of $\succeq$ to the subset of all $\mathcal{F}'$-adapted acts $h \in \mathcal{H}$ is continuous.[4]

Take two deterministic acts d and d'. The next axiom requires that if d is preferred to d' *unconditionally*, then d is also preferred to d' *conditional* on any event $A \in \cup_t \mathcal{F}_t$. That is, there are no taste shocks. In addition, the axiom requires that there is at least one event A such that both A and A^c preserve strict, as well as weak, rankings. The latter is a technical requirement which roughly posits that the probability of at least one event is bounded away from 0 and 1. To state the axiom formally, given $h \in \mathcal{H}, d \in X^\infty, t \in T$, and $A \in \mathcal{F}_t$, let $dA_t h$ denote the act $g \in \mathcal{H}$ such that $g_k(\omega) = d_k$

[4]Finite Continuity is weaker than the more traditional requirement that $\succeq$ be continuous. To illustrate the difference, suppose that $\succeq$ has an expected utility representation as in (1.1). Requiring $\succeq$ to be continuous would imply that the measure $p \in \Delta(\Omega)$ is countably additive. Following Savage [30], it is common in axiomatic work to require that beliefs are only finitely additive. FC makes this possible.

for all $k \geq t$ and all $\omega \in A$, and $g_k(\omega') = h_k(\omega')$ otherwise. Thus, $dA_t h$ is the act obtained from h by replacing all outcomes in states $\omega \in A$ and periods $k \geq t$ with the corresponding outcomes of the act d.

State Independence (SI): For all $t \in T, A \in \mathcal{F}_t$, and acts $h \in \mathcal{H}, d, d' \in X^\infty$ such that $h_k = d_k = d'_k$ for all $k \leq t-1$, if $d \succeq d'$, then $dA_t h \succeq d'A_t h$. In addition, there is some $t \in T$ and $A \in \mathcal{F}_t$ such that if $h_k = d_k = d'_k$ for all $k \leq t-1$ and $d \succ d'$, then $dA_t h \succ d'A_t h$ and $dA_t^c h \succ d'A_t^c h$.

Following the literature, we call an event $A \in \cup_t \mathcal{F}_t$ with the properties posited by the second part of State Independence **essential**. In the context of the expected utility model in (1.1), an event A is essential if and only if $p(A) \in (0,1)$, that is, if neither A nor A^c are assigned probability zero. In the context of the maxmin model in (1.2), an event A is essential if and only if $p(A) \in (0,1)$ for every $p \in P$.

The next axiom is central to our analysis. It extends Koopmans' [21] notion of stationarity from deterministic to stochastic environments. Such an extension was first considered in Epstein [9], albeit in a different setting, and later on in Kochov [19].[5] To understand what it implies for choice under uncertainty, note that an event $A \in \mathcal{F}_t$ may affect consumption in a more distant period $t+k$. The extension requires that, no matter how distant the consequences, the individual exhibits the same degree of uncertainty aversion. An example is given in Figure 2 in Section 7.3. To state the axiom formally, let (x,h) denote the act $g \in \mathcal{H}$ such that $g_0 = x$ and $g_t = h_{t-1}$ for all $t > 0$, where $x \in X$ and $h \in \mathcal{H}$. Thus, the act (x,h) is obtained from h by postponing the consumption date of each outcome of h by one period and inserting x in period $t = 0$. Note also that because the underlying filtration $\mathcal{F}$ remains fixed, this transformation does not alter the dates at which the relevant uncertainty resolves. The axiom requires that this transformation has no effect on preferences.

Path Stationarity (PS): For all acts $h, g \in \mathcal{H}$ and outcomes $x \in X$, $h \succeq g$ if and only if $(x,h) \succeq (x,g)$.

If we restrict PS to the space X^∞ of deterministic acts, we obtain Koopmans' [21] stationarity axiom, which requires that for all $x \in X$ and $d, d' \in X^\infty$, $d \succeq d'$ if and only if $(x,d) \succeq (x,d')$. As we claimed earlier, PS is thus an extension of Koopmans' axiom. To simplify the exposition, from now on we say that a preference relation

[5] There are several ways by which one can extend Koopmans' notion of stationarity from deterministic to stochastic environments. We compare two of these notions in Section 7.3.

$\succeq$ on $\mathcal{H}$ is **path stationary** if it satisfies FC, SI, and PS. Similarly, a preference relation $\succeq$ on X^∞ is **stationary** if it is continuous in the product topology on X^∞ and stationary in the sense of Koopmans [21].

For future reference, we introduce two more axioms, which are implied by Path Stationarity and State Independence. The first is **History Independence**. It requires that for all $x, y \in X, h, g \in \mathcal{H}$, $(x, h) \succeq (x, g)$ if and only if $(y, h) \succeq (y, g)$. Thus, if two acts yield an identical outcome in period $t = 0$, their ranking is independent of that outcome.

To state the second axiom, let $h(\omega) := (h_0(\omega)h_1(\omega), ...) \in X^\infty$ be the consumption stream delivered by an act $h \in \mathcal{H}$ in state ω. A preference relation $\succeq$ on $\mathcal{H}$ satisfies **Monotonicity** if for all $h, g \in \mathcal{H}$, $h \succeq g$ whenever $h(\omega) \succeq g(\omega)$ for every $\omega \in \Omega$. Thus, h is preferred to g whenever it yields a better consumption stream in every state of the world.[6]

If we abstract from the second part of State Independence, which required the existence of an essential event, then Monotonicity is strictly stronger than State Independence. As we explain in greater detail in Bommier et al. [4], the main additional implication of Monotonicity is that the individual evaluate all uncertainty through its impact on lifetime utility. To understand this, consider two acts $h, g \in \mathcal{H}$ such that $h(\omega) \sim g(\omega)$ for every $\omega \in \Omega$. That is, state by state the two acts yield identical levels of lifetime utility. The intertemporal distribution of risk can be very different however. For example, h could yield uncertain outcomes in a single period only, while g could yield uncertain outcomes in every period $t > 0$. By Monotonicity, such differences do not matter as long as the acts h and g yield the same level of lifetime utility in every state $\omega \in \Omega$.

3.2 A Representation Theorem

Let $\succeq$ be a path stationary preference relation on $\mathcal{H}$ and $U : X^\infty \to \mathbb{R}$ a continuous function representing the ranking of deterministic consumption streams. Given U, every act $h \in \mathcal{H}$ can be converted into a random variable in B^0 by associating each state $\omega \in \Omega$ with the utility of the consumption stream $h(\omega) \in X^\infty$. Denote this random variable by $U \circ h \in B^0$. It measures the lifetime utility induced by h state by

[6] It is obvious that Path Stationarity implies History Independence. That Path Stationarity and State Independence imply Monotonicity is shown in Lemma 7 in the appendix.

state. Given the function $U : X^\infty \to \mathbb{R}$ and a certainty equivalent $I : B^0 \to \mathbb{R}$, let

$$V(h) := I(U \circ h) \quad \forall h \in \mathcal{H}. \tag{3.1}$$

Say that the pair (U, I) **represents** a preference relation $\succeq$ on $\mathcal{H}$ if the function $V : \mathcal{H} \to \mathbb{R}$ represents $\succeq$.

To state our first theorem, suppose $U : X^\infty \to \mathbb{R}$ is a nonconstant function taking the form

$$U(x_0, x_1, \ldots) = u(x_0) + b(x_0)u(x_1) + b(x_0)b(x_1)u(x_2) + \ldots$$

where $u : X \to \mathbb{R}$ and $b : X \to (0,1)$ are continuous functions. In such a case, we say that U is an **Uzawa-Epstein utility function** and write (u, b) to denote it. If a preference relation $\succeq$ on X^∞ admits an Uzawa-Epstein utility function, we say that $\succeq$ is an **Uzawa-Epstein preference relation**. Finally, a certainty equivalent $I : B^0 \to \mathbb{R}$ is **regular** if there is some event $A \in \cup_t \mathcal{F}_t, A \notin \{\Omega, \varnothing\}$, such that I is strictly increasing when restricted to the space of $\{A, A^c\}$-measurable functions $\xi \in B^0$.[7]

Theorem 1 *A preference relation $\succeq$ on $\mathcal{H}$ is path stationary if and only if it has a representation (U, I) such that $U : X^\infty \to \mathbb{R}$ is an Uzawa-Epstein utility function (u, b) and the certainty equivalent I is regular, translation invariant and $b(x)$-homogeneous for every $x \in X$. Furthermore, if $b(x) \neq b(y)$ for some $x, y \in X$, then I is positively homogeneous.*

Theorem 1 can be viewed as a generalization of a representation result in Epstein [9] in that it delivers an Uzawa-Epstein utility $U : X^\infty \to \mathbb{R}$ without the additional assumption that uncertainty be evaluated in accordance with expected utility, which in the present context would mean that $\succeq$ have a representation (U, I) such that $I(\xi) = \mathbb{E}_p \xi$ for some probability measure $p \in \Delta(\Omega)$. Theorem 1 can also be viewed as a generalization of a representation result in Kochov [19] in that it delivers a translation invariant and positively homogeneous certainty equivalent I without the additional assumption that $\succeq$ have a representation (U, I) such that U is additively separable across time.

From now on, we write (u, b, I) for the utility representation deduced in Theorem 1.

[7]Regularity is needed to account for the existence of an essential event A, which is posited by State Independence.

4 Ambiguity or Endogenous Discounting?

The standard model of intertemporal choice implies that individuals do not care about the autocorrelations in their consumption profile. Indeed, note that

$$\mathbb{E}_p[\sum_t \beta^t u \circ h_t] = \sum_t \beta^t \mathbb{E}_p[u \circ h_t].$$

From the expression on the right, we see that to compute the utility of an act h, it is enough to know the marginal distribution of consumption within each period t: given these marginals, the autocorrelations do not matter. By comparison, the next axiom captures an individual who is averse to positive autocorrelation. First, given $t \geq 0$, $d = (x_0, x_1, ...) \in X^\infty$, and an act $h \in \mathcal{H}$, let $(d_{-(t,t+1)}, h_t, h_t)$ denote the act $\hat{h} \in \mathcal{H}$ such that $\hat{h}_k = x_k$ for all $k \notin \{t, t+1\}$ and $\hat{h}_t = h_t$ and $\hat{h}_{t+1} = h_t$. Given acts $h, g \in \mathcal{H}$, an act $(d_{-(t,t+1)}, h_t, g_t)$ can be similarly defined.

Intertemporal Hedging Against Uncertainty in Levels [IHUL]: For all $t \in T, d \in X^\infty$, and $h, g \in \mathcal{H}$,

$$(d_{-(t,t+1)}, h_t, h_t) \sim (d_{-(t,t+1)}, g_t, g_t) \quad \text{implies that} \quad (d_{-(t,t+1)}, h_t, g_t) \succeq (d_{-(t,t+1)}, g_t, g_t).$$

To understand the axiom, consider the act $(d_{-(t,t+1)}, g_t, g_t)$ and note that its outcomes in periods t and $t+1$ are perfectly positively correlated. Such correlation amplifies the uncertainty faced in either period: a bad outcome in period t implies a bad outcome in period $t+1$. An individual may prefer the act $(d_{-(t,t+1)}, h_t, g_t)$ whereby "different bets are taken in different periods." This act affords a form of time diversification which reduces the overall uncertainty faced by the individual.

Kochov [19] showed that such intertemporal hedging is indicative of ambiguity aversion provided that the ranking of deterministic consumption streams is time separable. The latter assumption is not innocuous. It is known from Epstein [9] that an expected utility model with endogenous discounting, such as the one in (1.1), exhibits intertemporal hedging if $b(x) < b(y)$ for all $x, y \in X$ such that $U(x, x, ...) > U(y, y, ...)$. This condition, known as **increasing marginal impatience**, is especially common in applied work where it insures the existence and stability of steady states.[8] Intertemporal hedging, of the type postulated by IHUL, can thus arise for two distinct reasons: one is ambiguity aversion, the other is an intertemporal complementarity in the individual's tastes.

[8] See Lucas and Stokey [24], Epstein [10], Backus et al. [3].

The goal of this section is to identify a type of intertemporal hedging that is indicative of ambiguity aversion even once we allow for endogenous discounting. Somewhat curiously, the pursuit of this goal takes us through a notion of impatience introduced by Koopmans [21]. First, given $t > 0, a, b \in X^t$, and $d \in X^\infty$, let (a, b, d) denote the consumption stream $(x_0, x_1, ...) \in X^0$ such that $(x_0, ..., x_{t-1}) = a, (x_t, ...x_{2t-1}) = b$, and $(x_{2t}, x_{2t+1}, ...) = d$. Given $a \in X^t$, a stream $(a, a, ...) \in X^\infty$ is similarly defined.

Impatience: For all $t > 0, a, b \in X^t$, and $d \in X^\infty$,

$$(a, a, a, ...) \succeq (b, b, b, ...) \quad \text{if and only if} \quad (a, b, d) \succeq (b, a, d).$$

Impatience says that the individual wants to consume first the outcome which he likes better. Interestingly, Koopmans [21] conjectured that all stationary preferences on X^∞ satisfy Impatience. While this conjecture was proved wrong by Koopmans et al. [22], Epstein [9] observed that every Uzawa-Epstein preference relation on X^∞ satisfies Impatience. With this in mind, Theorem 1 delivers an analogue of Koopmans' conjecture: if there is uncertainty, every path stationary preference relation satisfies Impatience.

To see why Impatience is important to us, recall from Section 3.1 that Path Stationarity implies History Independence. Hence, current consumption does not affect how the individual ranks future outcomes. An intertemporal complementarity in the tastes of the individual may therefore arise only when the anticipation of future outcomes affects current choices, that is, only when there is a form of *future dependence*.[9] But note that Impatience restricts the future dependencies that may arise. In particular,

$$(a, b, d) \succeq (b, a, d) \quad \text{if and only if} \quad (a, b, d') \succeq (b, a, d') \tag{4.1}$$

for all $t > 0, a, b \in X^t$, and $d, d' \in X^\infty$. Thus, when the individual is asked about the order in which he wants to consume a and b, his answer does not depend on the continuation stream d.

The preceding discussion suggests immediately how IHUL should be modified. Once again, the individual would seek to take different bets in different time periods. However, the uncertainty he would try to hedge would no longer concern the *level* of consumption in a given time period t. Instead, he would try to hedge uncertainty

[9] A **future dependence** arises if there are $t > 0, a, b \in X^t$, and $d, d' \in X^\infty$ such that $(a, d) \succeq (b, d)$ while $(a, d') \prec (b, d')$.

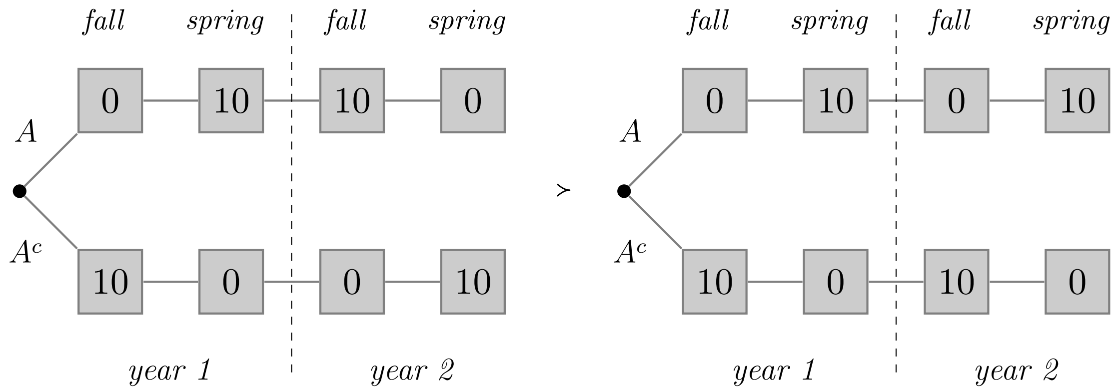

Figure 1: Intertemporal hedging when the relevant uncertainty concerns the timing of a given outcome.

about the *order* in which a given set of outcomes are consumed. As we just argued, in such situations there are no intertemporal complementarities that stem from the individual's tastes. Intertemporal hedging, which is in itself a type of intertemporal complementarity, can therefore arise only from the individual's attitudes toward uncertainty.

To give an example, consider Figure 1 in which time spans a period of two years, with each year consisting of a fall and a spring semester. A payment of *$10* is received in each year but the exact timing - fall or spring semester - depends on the realization of an event A. On the left, the time of delivery in year 1 is perfectly negatively correlated with the time of delivery in year 2. On the right, the correlation is positive. To abstract from anything other than these autocorrelations, suppose further that the individual views the events A and A^c as equally likely. One can then interpret the ranking in Figure 1 as indicative of an individual who prefers negative to positive correlation. As before, positive correlation is "bad" because it amplifies the uncertainty faced in any given year: a late (early) payment in year 1 implies a late (early) payment in year 2, with the end result that ex ante, lifetime utility becomes more dispersed. Theorem 2 below confirms this interpretation by showing that the ranking in Figure 1 is indeed driven by the individual's attitudes toward uncertainty. In particular, the ranking reveals an individual who is concerned about ambiguity.[10]

[10] Two simplifying assumptions are made in this example: there is no consumption in period $t = 0$ and the events A and A^c are viewed as equally likely. The formal axiom does not require these assumptions. Also, the payments in the example should be thought of as consumption levels: our

To proceed more formally, fix some $t > 1$ and a finite stream $a := (x_0, x_1, ..., x_{t-1}) \in X^t$ of outcomes. For every permutation $\pi : \{0, 1, ..., t-1\} \to \{0, 1, ..., t-1\}$, let πa denote the stream of outcomes obtained from $a \in X^t$ by permuting the order of its elements according to π. That is, $\pi a = (x_{\pi(0)}, x_{\pi(1)}, ..., x_{\pi(t-1)}) \in X^t$. Say that $h \in \mathcal{H}$ is a **repeating permutation act** if there is some $t \in T$ and $a \in X^t$, and for every $\omega \in \Omega$, a permutation $\pi_\omega : \{0, 1, ..., t-1\} \to \{0, 1, ..., t-1\}$ such that $h(\omega) = (\pi_\omega a, \pi_\omega a, ...)$ for every $\omega \in \Omega$. Repeating permutation acts have two special features. First, all outcomes are drawn from the elements of a. The uncertainty is about the order in which these outcomes will be consumed - a fact captured by the state-dependent permutations π_ω. Second, such acts have a repeating structure in that if one order (permutation) prevails during the first t periods, the same order will prevail in all subsequent blocks of t periods. This repeating structure means that uncertainty is compounded: If an event A obtains in which the outcomes in the first block are ordered unfavorably, then the same unfavorable order prevails in all subsequent periods. The act on the right hand side of Figure 1 is an example of a repeating permutation act.[11]

Confronted with a repeating permutation act, an individual may choose to break the perfect positive correlation across blocks. In particular, take some $t \in T$ and $a \in X^t$. Let $h, g \in \mathcal{H}$ be two repeating permutation acts that differ in the way the elements of a are ordered. Thus, for every $\omega \in \Omega$, $h(\omega) = (\pi_\omega a, \pi_\omega a, ...)$ and $g(\omega) = (\tilde{\pi}_\omega a, \tilde{\pi}_\omega a, ...)$ but the permutations $\pi_\omega, \tilde{\pi}_\omega$ are potentially different. If $h \sim g$, the logic of intertemporal hedging suggests that the individual would prefer the act $m \in \mathcal{H}$ such that $m(\omega) = (\pi_\omega a, \tilde{\pi}_\omega a, \tilde{\pi}_\omega a, ...)$ for every $\omega \in \Omega$. Such an act m appears on the left hand side of Figure 1. Below, we state a stronger axiom whereby the individual may use any act $g \in \mathcal{H}$ to hedge the uncertainty in a repeating permutation act $h \in \mathcal{H}$. To understand the axiom, it is helpful to think of an act $h \in \mathcal{H}$ as a function from Ω into X^∞.

Intertemporal Hedging against Uncertainty in Timing [IHUT]: Take a per-

focus is on direct preferences over consumption streams, not on indirect preferences over income streams.

[11] Repeating acts are often useful in the study of path stationary preferences. For a different application, see Kochov [20].

mutation act $h \in \mathcal{H}$ and an act $g \in \mathcal{H}$ of the form

$$h := \begin{cases} (\pi_1 a, \pi_1 a, ...) & \text{if } \omega \in A_1 \\ \dots & \\ (\pi_n a, \pi_n a, ...) & \text{if } \omega \in A_n \end{cases} \quad \text{and} \quad g := \begin{cases} d_1 & \text{if } \omega \in A_1 \\ \dots & \\ d_n & \text{if } \omega \in A_n. \end{cases}$$

Above, $\{A_1, ..., A_n\} \subset \cup_t \mathcal{F}_t$ is a partition of Ω, $a \in X^t$ for some $t > 1$, $d_i \in X^\infty$ for every $i \in \{1, 2, ..., n\}$, and each $\pi_i : \{0, 1, ..., t-1\} \to \{0, 1, ..., t-1\}$ is a permutation. If $h \geq g$, then

$$m := \begin{cases} (\pi_1 a, d_1) & \text{if } \omega \in A_1 \\ \dots & \\ (\pi_n a, d_n) & \text{if } \omega \in A_n \end{cases} \quad \geq \quad g := \begin{cases} d_1 & \text{if } \omega \in A_1 \\ \dots & \\ d_n & \text{if } \omega \in A_n \end{cases}.$$

We are ready to state the main theorem of this section.

Theorem 2 *A path stationary preference relation $\geq$ on $\mathcal{H}$ satisfies IHUT if and only if it has a representation (u, b, I) such that*

$$I(\xi) = \min_{p \in P} \mathbb{E}_p \xi$$

for some weak-closed and convex set P of probability measures on $(\Omega, \cup_t \mathcal{F}_t)$. Moreover, the set P is nonsingleton if and only if for some acts h, g, and m, as in the statement of IHUT, we have $m \succ g$.*

5 Uniqueness of the Representations

This section investigates the uniqueness of the representation (u, b, I) derived in Theorem 1. First, consider a preference relation $\geq$ on X^∞ with two Uzawa-Epstein representations.

Theorem 3 *Suppose a preference relation $\geq$ on X^∞ has two Uzawa-Epstein representations:*

$$U(x_0, x_1, x_2, ...) = u(x_0) + b(x_0)u(x_1) + b(x_0)b(x_1)u(x_2) + ...$$

$$\hat{U}(x_0, x_1, x_2, ...) = \hat{u}(x_0) + \hat{b}(x_0)\hat{u}(x_1) + \hat{b}(x_0)\hat{b}(x_1)\hat{u}(x_2) + ...$$

Then, $b = \hat{b}$ and $U = \alpha \hat{U} + \gamma$ for some $\alpha \in \mathbb{R}_{++}, \gamma \in \mathbb{R}$.

Theorem 3 strengthens a uniqueness result in Epstein [9]. The latter is derived for the expected utility model in (1.1). In particular, the expected utility hypothesis implies directly that the function $U : X^\infty \to \mathbb{R}$ is cardinally unique. Using this, Epstein [9] goes on to prove that $b = \hat{b}$. Theorem 3 is stronger in that it delivers the same conclusions by taking as primitive only the ranking of deterministic consumption streams. The proof is also different in that we first establish the uniqueness of the function $b : X \to (0,1)$ and then use this to prove that the function $U : X^\infty \to \mathbb{R}$ is cardinally unique.

The uniqueness of the function $b : X \to (0,1)$ suggests that b has a well defined ordinal meaning. This meaning can be made clear if one assumes that $X \subset \mathbb{R}$ and utility is suitably differentiable. Then, given a constant stream $(x, x, ...) \in X^\infty$, $b(x)^{-1}$ is equal to the marginal rate of substitution between consumption in any two periods t and $t+1$. Put differently, $1 - b(x)^{-1}$ is the rate of time preference along the constant stream $(x, x, ...) \in X^\infty$. See Epstein [9, p.137] for details.

Suppose now that $\succeq$ is a preference relation on $\mathcal{H}$ with a representation (u, b, I) as in Theorem 1. Building on Theorem 3, our next result shows that the certainty equivalent I is unique in two cases. The first is when discounting is endogenous, that is, when $b(x) \neq b(y)$ for some $x, y \in X$. If discounting is exogenous, the certainty equivalent is unique if we assume IHUT as well.

Corollary 4 *Suppose a path stationary preference relation $\succeq$ on $\mathcal{H}$ has two representations (u, b, I) and $(\hat{u}, \hat{b}, \hat{I})$ as in Theorem 1. From Theorem 3, we know that $b = \hat{b}$. If the function $b : X \to (0,1)$ is nonconstant, then $I = \hat{I}$. The same is true if b is constant and $\succeq$ satisfies IHUT.*

6 A Converse of Theorem 3

Cardinally unique representations, such as the one obtained in Theorem 3, are helpful in many contexts. It may thus be of interest to find a converse of Theorem 3, that is, to identify all stationary preferences on X^∞ that have a cardinally unique representation.

Given a stationary preference relation $\succeq$ on X^∞, let $\mathcal{U}_\succeq$ be the space of all continuous functions $U : X^\infty \to \mathbb{R}$ representing $\succeq$ and let $\mathcal{U}^* \subset \mathcal{U}_\succeq$ be the subset that takes the Uzawa-Epstein form. Note that the set $\mathcal{U}^*$ could be empty, in which case $\succeq$ has no Uzawa-Epstein utility. Next, say that a set $\mathcal{U}' \subset \mathcal{U}_\succeq$ is **cardinal** if for any two functions $U, U' \in \mathcal{U}'$, there are $\alpha \in \mathbb{R}_{++}$ and $\gamma \in \mathbb{R}$ such that $U = \alpha U' + \gamma$. Note that having a cardinal set $\mathcal{U}'$ of utility functions is a requirement that, by itself, can be trivially satisfied. In particular, any singleton set $\mathcal{U}' \subset \mathcal{U}_{\succeq}$ is cardinal. Thus, cardinality is a more interesting property, the more functions the set $\mathcal{U}'$ contains. In the context of stationary preferences, a natural richness requirement on the set $\mathcal{U}'$ is that it inherits the recursive structure of the utility functions it contains. Formally, given a function $U \in \mathcal{U}_{\succeq}$ and any $x \in X$, define the function $U_x : X^\infty \to \mathbb{R}$ by letting $U_x(d) := U(x, d)$ for all $d \in X^\infty$. Then, say that the set $\mathcal{U}' \subset \mathcal{U}_{\succeq}$ is **recursive** if $U_x \in \mathcal{U}'$ for every $x \in X$ and $U \in \mathcal{U}'$. Since $\succeq$ is stationary, the set $\mathcal{U}_{\succeq}$ is recursive. So is the set $\mathcal{U}^*$. By Theorem 3, the set $\mathcal{U}^*$ is also cardinal. We want to know all stationary preferences $\succeq$ that admit a recursive, cardinal set $\mathcal{U}'$ of utility functions. The next result shows that only Uzawa-Epstein preferences do.

Corollary 5 *Let $\succeq$ be a stationary preference relation on X^∞ and $\mathcal{U}_{\succeq}$ the set of all continuous functions $U : X^\infty \to \mathbb{R}$ representing $\succeq$. Suppose $\mathcal{U}' \subset \mathcal{U}_{\succeq}$ is a nonempty, recursive, and cardinal set. Then, $\mathcal{U}' \subset \mathcal{U}^*$, that is, $\succeq$ is an Uzawa-Epstein preference relation.*

Even though the arguments are framed differently, this result is a corollary of Theorem 1 in Epstein [9].

7 Discussion

7.1 More On Intertemporal Hedging

Theorem 2 shows that within the class of path stationary preferences, IHUT is indicative of ambiguity aversion. It is important to note that the same is true more broadly. For example, it is known from Kochov [19] that the variational model of Maccheroni et al. [28] is not path stationary. Yet, the model satisfies IHUT and a strict ranking $m \succ g$ obtains for some acts h and g, except in the very special case in which the individual perceives no ambiguity and the model reduces to expected utility.

The variational model of Maccheroni et al. [28] is one in which the ranking of deterministic acts admits a utility function $U : X^\infty \to \mathbb{R}$ that is additively separable across time. As Kochov [19] shows, one can then use the simpler axiom IHUL to define ambiguity averse behavior. Thus, a more interesting question is whether IHUT is indicative of ambiguity aversion even when the function U is not additively separable and, in fact, when U is not of the Uzawa-Epstein kind. The ideas that led to the formulation of IHUT make it reasonable to conjecture that the answer is yes as long as History Independence and Impatience hold. We plan to explore this conjecture in future work.

In terms of the existing literature on ambiguity aversion, our main contribution is to identify what ambiguity aversion implies about *intertemporal* behavior. This question was first posed in Kochov [19]. We provide a more robust answer by relaxing the assumption that $U : X^\infty \to \mathbb{R}$ is time additive. It is also noteworthy that, as in Kochov [19], we carry out the analysis in a setting of purely subjective uncertainty. In particular, our paper differs from the seminal work of Gilboa and Schmeidler [18] in that we do not assume that the likelihoods of some events are objectively given. Of course, we are not the first to study ambiguity aversion in a purely subjective setting. Other papers that tackle this problem include Alon and Schmeidler [2], Casadesus-Masanell et al. [5], and Ghirardato and Marinacci [14], Ghirardato et al. [16]. Their approach is very different from ours however. Roughly, these papers begin by eliciting the individual's certainty equivalents for a suitably chosen family of acts.[12] Once these certainty equivalents are elicited, a wide range of choice comparisons can be used to elicit the ambiguity attitudes of the individual. By comparison, our notion of ambiguity aversion (IHUT) does not require the prior elicitation of certainty equivalents. In fact, IHUT can be formulated even when the outcome space X is discrete and certainty equivalents need not exist. To make this possible, we adopt a dynamic choice setting and focus on intertemporal behavior. The price we have to pay is that we need to distinguish ambiguity aversion from other phenomena that are specific to intertemporal choice, such as endogenous discounting. To draw such a distinction, our notion of ambiguity aversion ends up being based on a very narrow set of choice comparisons in which the individual tries to hedge a very specific type of uncertainty.

7.2 A Sketch of the Proof of Theorem 1

Theorem 1 and our results in Bommier et al. [4] utilize powerful but relatively unknown techniques from Lundberg's [26] work on functional equations. It may thus be helpful to sketch a proof of Theorem 1. We begin by recalling a result from Koopmans [21]. Let $U : X^\infty \to \mathbb{R}$ be a continuous utility function for a preference relation $\succeq$ on X^∞. Koopmans [21] shows that $\succeq$ is stationary if and only if U takes

[12] In the static setting of these papers, an **act** is a function $f : \Omega \to X$ from states into outcomes. A **certainty equivalent for an act** f is an outcome $x \in X$ such that $x \sim f$.

the recursive form

$$U(x_0, x_1, x_2, \ldots) = \phi(x_0, U(x_1, x_2, \ldots)), \quad \forall (x_0, x_1, \ldots) \in X^\infty, \tag{7.1}$$

for some function $\phi : X \times U(X^\infty) \to U(X^\infty)$ which is continuous in each argument and strictly increasing in the second. The function ϕ is commonly referred to as a **time aggregator** for the utility $U : X^\infty \to \mathbb{R}$. The name is motivated by the fact that ϕ computes ex ante utility as an average of current consumption and continuation utility.

Now, let $\succeq$ be a preference relation on $\mathcal{H}$ with a representation (U, I). Suppose U can be written recursively as in (7.1). It is then convenient to make the time aggregator ϕ explicit and write (U, ϕ, I) instead of (U, I). Given a random variable $\xi \in U \circ \mathcal{H} \subset B^0$, let $\phi(x, \xi)$ denote the function $\omega \mapsto \phi(x, \xi(\omega))$. Note that $\phi(x, \xi) \in B^0$ is a random variable. Take an act $(x, h) \in \mathcal{H}$ as in the statement of Path Stationarity and consider the equalities:

$$V(x, h) = I(\phi(x, U \circ h)) = \phi(x, I(U \circ h)), \quad \forall x \in X, h \in \mathcal{H}. \tag{7.2}$$

The first equality combines the definition of V with (7.1). The interesting equality is the second one. It says that there are two ways to compute the utility of an act (x, h). The expression $I(\phi(x, U \circ h))$ means that one first aggregates utility across time and then across states. Conversely, the expression $\phi(x, I(U \circ h))$ means that one first computes the expectation $I(U \circ h)$ of future utility and then aggregates across time by adding the utility of the initial outcome x. When these two computations agree, as they do in (7.2), we say that the certainty equivalent I and the time aggregator ϕ **permute**. The next lemma establishes a connection between (7.2) and Path Stationarity.

Lemma 6 *A preference relation $\succeq$ on $\mathcal{H}$ is path stationary if and only if it has a representation (U, ϕ, I) such that the time aggregator ϕ and the certainty equivalent I permute. The latter is true for all representations (U, ϕ, I) of a path stationary preference relation $\succeq$.*

Having proved Lemma 6, Lundberg's work comes into the picture. If the state space Ω consists of two elements only, that is, if $|\Omega| = 2$, then the second equality in (7.2) becomes what Lundberg [26] calls a distributivity equation. Lundberg shows that the solutions to such equations are well behaved locally, where *well behaved* means that, after a suitable monotone transformation, the functions (U, ϕ, I) have the properties sought after in Theorem 1. Lundberg's work leaves us with three problems we have to address.

First, Lundberg's results require a joint, technical restriction on the functions ϕ and I. We are able to show that this technical restriction is satisfied whenever these functions are part of a representation (U, ϕ, I) of a path stationarity preference relation. See Lemma 8 in Section A.1.1 of the appendix and how the lemma is applied in Section A.1.4.

Second, the local nature of Lundberg's solutions is a problem. We tackle this problem by showing that, under Path Stationarity, a well behaved *local solution* is enough to obtain a well behaved *global representation*, that is, a representation for the entire preference relation $\geq$ on $\mathcal{H}$. In particular, recall that $U \circ h$ denotes the function $\omega \mapsto U(h(\omega))$ and let $Im(U \circ h)$ denote its image. Suppose that (7.2) has a well behaved solution in some interval $O \subset \mathbb{R}$, by which we mean that the functions U, ϕ, I are well behaved whenever the outcomes $x \in X$ and the acts $h \in \mathcal{H}$ that appear in (7.2) are restricted so that $U(x, h), U(h), Im(U \circ h), Im(U \circ (x, h))$ are all within the set O. Take any two acts $h, h' \in \mathcal{H}$ and let $x^* \in X$ be such that $U(x^*, x^*, ...) \in O$. By Path Stationarity,

$$h \geq h' \Leftrightarrow (x^*, h) \geq (x^*, h') \Leftrightarrow (x^*, x^*, h) \geq (x^*, x^*, h') \Leftrightarrow \dots \tag{7.3}$$

As we increase the number of initial periods in which x^* is consumed, the acts in (7.3) converge to $(x^*, x^*, x^*, ...)$. Hence, their utilities are eventually contained in the set O in which the functions (U, ϕ, I) are well behaved. But the equivalences in (7.3) show that the restrictions of these functions to the set O represent the entire preference relation.

The final problem is that we have to solve the functional equation in (7.2) when Ω is arbitrary, not only when $|\Omega| = 2$. Though mostly technical, this extension of Lundberg's results, which we use in our companion paper Bommier et al. [4] as well, requires some effort.

7.3 Two Notions of Stationarity

There are several ways by which one can extend Koopmans' notion of stationarity from deterministic to stochastic environments. Figure 2 illustrates two of these extensions and the difference between them. The top part of the figure depicts Path Stationarity, while the bottom part depicts the second extension for which we reserve the name Stationarity. In both parts of the figure, the relevant uncertainty is the outcome of a coin toss. Consistent with the explanation in Section 3.1, Path Stationarity considers a situation in which the coin is flipped in period $t = 1$ always, but one varies the date on which consumption takes place. The axiom then requires that the individual's attitudes toward uncertainty do not depend on the date on which consumption takes place. The second extension, Stationarity, considers a situation in which one simultaneously changes the date on which the coin is flipped and the date on which consumption takes place. Remark furthermore that the time distance between when the coin is flipped and when consumption takes places is kept unchanged, while this is not the case for the upper part of Figure 2 which illustrates Path Stationarity.

Both extensions have appeared in the literature under the single name Stationarity. For example, Epstein [9] and Kochov [19] call Stationarity what we call Path Stationarity. It seems more typical, however, to reserve the name Stationarity for the extension in the bottom part of Figure 2. This is the terminology followed in Epstein and Zin [13] and Chew and Epstein [6], among others. To avoid confusion, we propose different names for the two extensions.[13]

The two extensions concern choice situations that are qualitatively different. Stationarity is of interest when the underlying uncertainty repeats itself. In such a case, an individual may confront the same uncertainty and decision problem at different moments in time, with Stationarity implying that the individual would make the same decision. In a dynamic setting, Stationarity implies that choices are history independent and time consistent. Path Stationarity is of interest when the impact of an event can be delayed over time. For example, consider an individual who has to choose between two jobs, each offering a career trajectory that depends on an event to be resolved today. Path Stationarity requires that the choice would be the same whether employment starts immediately after the event is resolved or after a one month vacation.

Given that there are two extensions, we have written two papers, one for each extension: Bommier et al. [4] study preferences that satisfy Monotonicity, Stationarity, and a standard recursivity assumption; the present papers studies preferences that satisfy State Independence and Path Stationarity.[14] Both Bommier et al. [4] and the present paper build on the work of Lundberg [26], but they do so in different ways.

[13]For each $x \in X$, let $T_x : X^\infty \to X^\infty$ be the mapping $d \mapsto (x, d)$ used to define stationarity in Koopmans' deterministic setting. To motivate the name *Path Stationarity*, note that in the study of stochastic processes, it is common to refer to $h(\omega) \in X^\infty$ as a possible *path* of the process $h \in \mathcal{H}$. and that the act $(x, h) \in \mathcal{H}$ in the statement of Path Stationarity is obtained from $h \in \mathcal{H}$ by applying T_x *path by path*.

[14]As was observed in Section 3.1, Path Stationarity and State Independence imply Monotonicity. In particular, Monotonicity is an axiom which we maintain in both papers.

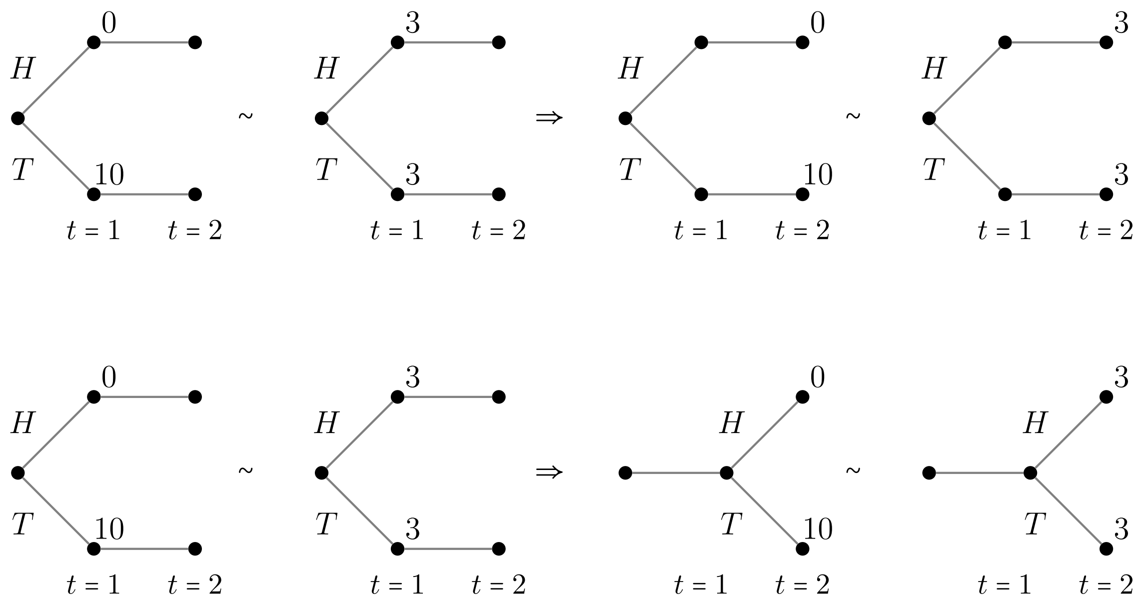

Figure 2: The top part of the figure depicts Path Stationarity. The bottom part depicts Stationarity. In both parts, the relevant uncertainty is the outcome of a coin toss. For the sake of simplicity, it is assumed that consumption takes place in a single period only.

As explained in Section 7.2, in this paper we use Path Stationarity to show that a technical restriction on the functions I and ϕ, which is required for Lundberg's results, is met, and that the local solutions in Lundberg [26] can be extended into a representation for the entire preference relation. In Bommier et al. [4], we do not assume Path Stationarity. Instead, we make an additional monotonicity assumption, Deterministic Monotonicity, which insures that the same technical restriction is met and that the local solutions in Lundberg [26] are in fact global.[15] Another difference is that in Bommier et al. [4], we have to solve a *system of generalized distributivity equations.* While Lundberg [26] provides solutions for such *generalized distributivity equations* as well, it is the specific structure of the *system* of equations implied by our axioms that narrows down the set of possible solutions and obtains sharp utility representations.

[15] Suppose $X \subset \mathbb{R}$ and that X is endowed with the usual linear order. **Deterministic Monotonicity** requires that $\succeq$ respect the implied pointwise order on X^∞.

7.4 Indifference to The Timing of Uncertainty

Related to the discussion in Section 7.3 is the Kreps and Porteus [23] notion of indifference to the timing of resolution of uncertainty, which we refer to as KPI. In direct contrast to Path Stationarity, KPI considers a situation in which one keeps the date of consumption fixed but varies the date on which uncertainty is resolved. In the context of Figure 2, KPI means that the indifference in the top right corner implies the indifference in the bottom right corner. While the axioms are conceptually different, there are settings in which Path Stationarity and KPI become equivalent restrictions on behavior. This is the case if we assume that uncertainty is IID over time and that preferences are recursive. On the other hand, Stationarity is not equivalent to KPI even under these additional assumptions. As is known from Strzalecki [31], the recursive variational preferences of Maccheroni et al. [28] are stationary but do not satisfy KPI.

To understand the role of IID uncertainty in the preceding paragraph, note that both Stationarity and KPI involve choice situations in which the resolution of uncertainty is postponed, or equivalently, in which the individual faces the same uncertainty in different periods. But in a setting of purely subjective uncertainty, with a fixed and arbitrary filtration $(\Omega, \{\mathcal{F}_t\}_t)$, the underlying uncertainty may never repeat itself, in which case Stationarity and KPI cannot be formulated.[16] Path Stationarity, on the other hand, can be formulated given any information structure $(\Omega, \{\mathcal{F}_t\}_t)$. At this level of generality, a link between Path Stationarity and KPI can no longer be made.

A Appendix

Given functions $f : X' \to Y'$ and $g : Y' \to Z'$, we use $g \circ f$ and gf interchangeably to denote the composition of the two functions. Given an interval $C \subset \mathbb{R}$, B^0_C denotes the sets of all functions $\xi \in B^0$ such that $\xi(\Omega) \subset C$. Given a finite algebra $\mathcal{F}' \subset \cup_t \mathcal{F}_t$, $B^0(\mathcal{F}')$ is the set of all $\mathcal{F}'$-measurable functions $\xi \in B^0$. The set $B^0_C(\mathcal{F}') \subset B^0(\mathcal{F}')$ is similarly defined. A function $I : B^0_C \to \mathbb{R}$ is **finite continuous** if for every finite algebra $\mathcal{F}' \subset \cup_t \mathcal{F}_t$, the function I is norm continuous when restricted to the set $B^0_C(\mathcal{F}') \subset B^0_C$.

[16] The simplest way to see this is to consider the extreme case in which all the uncertainty resolves in period $t = 1$ so that $\cup_t \mathcal{F}_t = \mathcal{F}_1$.

A.1 Proof of Theorem 1

Necessity of the axioms is obvious. We prove sufficiency.

A.1.1 Preliminaries

Lemma 7 *State Independence and Path Stationarity imply Monotonicity.*

Proof. Let $h, h' \in \mathcal{H}$ be such that $h(\omega) \geq h'(\omega)$ for every ω. Because h, h' are simple, there is some $t \in T$ such that h_k, h'_k are $\mathcal{F}_t$-measurable for every k. Fix some $a = (x_0, ..., x_{t-1}) \in X^t$ and consider the acts $(a, h), (a, h')$. By construction, $(a, h)(\omega) = (a, h(\omega))$. By PS, $(a, h(\omega)) \geq (a, h'(\omega))$ for every $\omega \in \Omega$. Moreover, $h \geq h'$ if and only if $(a, h) \geq (a, h')$, so it suffices to show the latter. Think of $(a, h), (a, h')$ as functions from Ω into X^∞. Both functions have finite range. Let $\{A_1, A_2, ..., A_n\}$ be the coarsest partition of Ω such that the functions are measurable.[17] Replace the infinite stream of (a, h') on A_1 by the respective infinite stream of (a, h). By SI, the new act is preferred to (a, h'). Take the new act and replace its infinite stream on A_2 by the respective infinite stream of (a, h). Apply SI again. After n such steps, we obtain (a, h). Because $\geq$ is transitive, $(a, h) \geq (a, h')$. ■

Lemma 8 *Consider a path stationary preference relation $\geq$ on $\mathcal{H}$. Then,*

1. *There are $x, y \in X, d \in X^\infty$ such that $(x, d) \succ (y, d)$.*
2. *For every $x \in X, h \in \mathcal{H}$, $(x, x, ...) \geq h$ if and only if $(x, h) \geq h$. Similarly, $h \geq (x, x, ...)$ if and only if $h \geq (x, h)$.*
3. *The best and worst sequences in X^∞ are constant. Denote them by $(z^*, z^*, ...)$ and $(z, z, ...)$.*
4. *Writing d^* for $(z^*, z^*, ...)$, we have $(z^*, z, d^*) \succ (z, z, d^*)$.*
5. *There exists a sequence $(x_n)_n$ in X, converging to z such that $(x_n, z, z, ...) \succ (z, z, z, ...)$. Moreover, for every $n \in \mathbb{N}, d \in X^\infty$, there is $d' \in X^\infty$ such that $(z, d') \sim (x_n, z, d)$.*

Proof. Property (1) is proved in Kochov [19, Lemma 5]. To prove (2), suppose $(x, x, ...) \geq h$, but $h \succ (x, h)$. Then, by PS, $h \succ (x, h) \succ (x, x, h)$. Repeating the argument, we obtain $h \succ (x^n, h)$ for every $n \in \mathbb{N}$, where x^n denotes the vector in

[17] Measurability with respect to a partition means measurability with respect to the algebra generated by the partition.

X^n all of whose components are equal to x. By Finite Continuity, $h \succ (x, x, ...)$, a contradiction. Analogous arguments establish the converse implication and the second equivalence postulated in (2). To prove (3), let $(z^*, z^*, ...)$ be the best among all constant sequences in X^∞. By Finite Continuity, it is enough to show that $(z^*, z^*, ...) \succeq (x_1, x_2, ..., x_n, z^*, z^*, ...)$ for all $(x_1, x_2, ..., x_n) \in X^n, n \in \mathbb{N}$. From (2), we know that $(z^*, z^*, ...) \succeq (x_k, z^*, z^*, ...)$ for every $k \leq n$. Making repeated use of this observation and Path Stationarity, we obtain

$$
\begin{aligned}
&(z^*, z^*, ...) \succeq (x_n, z^*, z^*, ...) \quad \Rightarrow \\
&(x_{n-1}, z^*, z^*, ...) \succeq (x_{n-1}, x_n, z^*, z^*, ...) \quad \Rightarrow \\
&(z^*, z^*, ...) \succeq (x_{n-1}, z^*, z^*, ...) \succeq (x_{n-1}, x_n, z^*, z^*, ...) \quad \Rightarrow \\
&... \\
&(z^*, z^*, ...) \succeq (x_1, x_2, ..., x_n, z^*, z^*, ...)
\end{aligned}
$$

Turn to (4). By way of contradiction, suppose $(z, z, d^*) \succeq (z^*, z, d^*)$. Since $d^* \succeq (z, d^*)$ and $\succeq$ is path stationary, we obtain

$$(z, d^*) \succeq (z, z, d^*) \succeq (z^*, z, d^*).$$

By PS, $(z^*, z, d^*) \succeq (z^*, z^*, z, d^*)$. By the contradiction hypothesis,

$$(z, z, d^*) \succeq (z^*, z, d^*) \succeq (z^*, z^*, z, d^*). \tag{A.1}$$

Since $(z, d^*) \succeq (z^*, z, d^*)$,

$$(z^*, z^*, z, d^*) \succeq (z^*, z^*, z^*, z, d^*). \tag{A.2}$$

Combining (A.1) and (A.2), we get

$$(z, z, d^*) \succeq (z^*, z^*, z^*, z, d^*).$$

Repeating the argument, we get $(z, z, d^*) \succeq ((z^*)^n, z, d^*)$ for every $n \in \mathbb{N}$. By Finite Continuity , $(z, z, d^*) \succeq d^*$, contradicting property (2). Finally, turn to property (5). From (2), we know that $(z^*, z, z, ...) \succ (z, z, ...)$. Because X is connected, there is a sequence $(x_n)_n$ in X converging to z such that $(x_n, z, z, ...) \succ (z, z, ...)$. By (2) again, $(z, d^*) \succ (z, z, d^*)$. Since (x_n, z, d^*) converges to (z, z, d^*) as $n \to \infty$, we have $(z, d^*) \succ (x_n, z, d^*)$ for all n larger than some $N \in \mathbb{N}$. By construction, it is also the case that

$$(x_n, z, d^*) \succeq (x_n, z, z, z, ...) \succ (z, z, z, ...) \quad \forall n \in \mathbb{N}.$$

Combining the last two observations gives $(z,d^*) \succ (x_n,z,d^*) \succ (z,z,z,...)$ for all $n \geq N$. By PS, it is also the case that

$$(x_n,z,d^*) \succeq (x_n,z,d) \succeq (x_n,z,z,...) \succ (z,z,...).$$

for all $d \in X^\infty, n \in \mathbb{N}$. Summing up, for every $n \geq N$ and $d \in X^\infty$, we have

$$(z,d^*) \succ (x_n,z,d) \succ (z,z,z,...)$$

By Finite Continuity and the connectedness of X^∞, there is $d' \in X^\infty$ such that $(z,d') \sim (x_n,z,d)$. ■

A.1.2 Proof of Lemma 6

We begin by proving an analogue of Lemma 6 whereby a representation (U,ϕ,I) is defined by using a more permissive notion of a certainty equivalent. In this more permissive sense, (U,ϕ,I) is a **representation** for a path stationary preference relation $\succeq$ on $\mathcal{H}$ if (i) $U : X^\infty \to \mathbb{R}$ is continuous, (ii) $\phi : X \times U(X^\infty) \to \mathbb{R}$ is a time aggregator for U, that is, U and ϕ satisfy the recursion in (7.1), and (iii) $I : B^0_{U(X^\infty)} \to \mathbb{R}$ is a normalized, increasing, and finite continuous function. The latter notion of a certainty equivalent is weaker than the one given in Section 2 in that I is defined only on $B^0_{U(X^\infty)}$, rather than on the entire space B^0, and I is only finite continuous rather than norm continuous.

Because X^∞ is connected and separable in the product topology, we know from Debreu [7] that there is a continuous function $U : X^\infty \to \mathbb{R}$ that represents the restriction of $\succeq$ to X^∞. For every $x \in X$ and $k \in U(X^\infty)$, define $\phi(x,k) := U(x,d_k)$ where $d_k \in X^\infty$ is such that $U(d_k) = k$. So defined, ϕ is continuous in its first argument since U is continuous. Because the restriction of $\succeq$ to X^∞ is stationary, ϕ is strictly increasing in the second argument. Since U is continuous, the set $\phi(x,U(X^\infty))$ is connected for every $x \in X$. Conclude that ϕ is continuous in the second argument and, ultimately, that ϕ is a time aggregator for U. Turn to the construction of a certainty equivalent $I : B^0_{U(X^\infty)} \to \mathbb{R}$. Since X^∞ is connected, a standard argument shows that for every act $h \in \mathcal{H}$, there is an act $d_h \in X^\infty$ such that $h \sim d_h$. Extend U from X^∞ to $\mathcal{H}$ by letting $V(h) := U(d_h)$. Recall that $U \circ h$ denotes the function $\omega \mapsto U(h(\omega))$ and let $U \circ \mathcal{H} := \{U \circ h : h \in \mathcal{H}\} \subset B^\circ$. Define $I : U \circ \mathcal{H} \to \mathbb{R}$ by letting $I(U \circ h) = V(h)$. By Lemma 7, $\succeq$ satisfies Monotonicity. It follows that I is well defined and increasing. Because $\succeq$ satisfies FC, I is finite continuous. By definition, $I(k) = k$ for all $k \in U(X^\infty)$, that is, I is normalized. Altogether, (U,ϕ,I) is a representation of $\succeq$.

The next step is to show that for all representations (U, ϕ, I) of $\succeq$, I and ϕ permute. Fix some $x \in X, h \in \mathcal{H}$, and note that $U \circ (x, h) = \phi(x, U \circ h)$. Choose $d \in X^\infty$ such that $d \sim h$. By Path Stationarity, $(x, d) \sim (x, h)$. Since the certainty equivalent I is normalized,

$$I(\phi(x, U \circ h)) = U(x, d) = \phi(x, U(d)) = \phi(x, I(U \circ h)).$$

Thus, I and ϕ permute. That $\succeq$ must be path stationary $\succeq$ if it has a representation such that I and ϕ permute is obvious.

We have thus proved the desired analogue of Lemma 6. It is notable that this analogue of Lemma 6 is what we need in the rest of the proof of Theorem 1. To prove Lemma 6 as stated in Section 7.2, the only additional thing we have to prove is the existence of a representation (U, ϕ, I) in which the certainty equivalent I is defined on B^0 and is norm continuous. This will follow from Theorem 1, once its proof is completed. See Lemmas 17 and 24.[18]

A.1.3 Iteration Groups

We need to introduce some mathematical concepts from Lundberg [26]. Let $C \subset \mathbb{R}$ be a nonempty, open interval and λ an extended real number in $\mathbb{R}_{++} \cup \{+\infty\}$. Let $\{g^\alpha : \alpha \in (-\lambda, \lambda)\}$ be a family of functions such that each function g^α is defined on an interval $C^\alpha \subset C$ and $g^\alpha(C^\alpha) \subset C$. Suppose each function g^α is continuous and strictly increasing, and its graph disconnects the Cartesian product C^2.[19] Suppose further that the graph of $g^\alpha \circ g^\beta$ is a subset of the graph of $g^{\alpha+\beta}$, with the latter holding for all $\alpha, \beta \in (-\lambda, \lambda)$ such that $\alpha + \beta \in (-\lambda, \lambda)$. We call such a family of functions an **iteration group over** C.

Whenever an iteration group $\{g^\alpha : \alpha \in (-\lambda, \lambda)\}$ over an interval C is given, we assume that the group is **maximal**, that is, there is no other iteration group $\{\tilde{g}^\alpha : \alpha \in (-\tilde{\lambda}, \tilde{\lambda})\}$ over C such that $\lambda < \tilde{\lambda}$ and $g^\alpha = \tilde{g}^\alpha$ for all $\alpha \in (-\lambda, \lambda)$. When no confusion arises, we may also suppress the interval C and the bound λ, and speak simply of an iteration group $\{g^\alpha\}$.

[18] In terms of the logic of the proof of Theorem 1, it would have been better to state Lemma 6 by using the more permissive notion of a certainty equivalent. However, we wanted to avoid introducing two different notions of a certainty equivalent, which differ in some technical aspects only, in the main text.

[19] The graph of g^α disconnects C^2 if for every $(x, y), (x', y') \in C^2$ such that $x, x' \in C^\alpha$ and $y > g^\alpha(x), y' < g^\alpha(x')$, a continuous curve in C^2 that connects (x, y) to (x', y') must intersect the graph of g^α.

Let $j : C \to C$ be the identify function on C and $\{g^\alpha\}$ an iteration group on C. Then, $g^0 = j$. More generally, if α is an integer, then g^α is the α-iterate of the function g^1, with the domain of the α-iterate restricted so that the range does not exceed the set C. An iteration group $\{g^\alpha\}$ is thus a way of defining the α-iterate of a function g^1 for *all* α, integer or not, in a manner that coincides with the usual definition when α is an integer.

An iteration group $\{g^\alpha : \alpha \in (-\lambda, \lambda)\}$ over C induces an iteration group over any nonempty, open interval $D \subset C$. In particular, let $\hat{\lambda}$ be the supremum of all $\alpha \in (-\lambda, \lambda)$ such that the graph of $g^\alpha|_D$ intersects D^2. Then, $\{g^\alpha|_D : \alpha \in (-\hat{\lambda}, \hat{\lambda})\}$ is an iteration group over D.

To give an example of an iteration group, let $C = \mathbb{R}$, $\lambda = +\infty$. Let $f(k) = a+bk, a, b \in \mathbb{R}$ be an affine function defined for all $k \in \mathbb{R}$. Assume that $b \neq 1$. Then, there is a unique iteration group $\{g^\alpha\}$ such that $g^1 = f$. In fact, we can compute each function g^α explicitly:

$$g^\alpha(k) = a\frac{1 - b^\alpha}{1 - b} + b^\alpha k. \tag{A.3}$$

In this example, all functions g^α, $\alpha \neq 0$, share the same fixed point $k^* = a(1 - b)^{-1}$. This observation holds more generally. Thus, if $\{g^\alpha\}$ is an iteration group over C and $g^\alpha(k) = k$ for some $\alpha \neq 0$ and $k \in C$, then $g^\beta(k) = k$ whenever k is in the domain of g^β.

Say that an iteration group $\{g^\alpha\}$ is **fixed point free** if none of the functions g^α, $\alpha \neq 0$, has a fixed point. As observed in Lundberg [26], we can then find a function $L : C \to \mathbb{R}$ such that $g^\alpha(k) = L^{-1}(L(k) + \alpha)$ for all k in the domain of g^α and all $\alpha \in (-\lambda, \lambda)$. The function L is called an **Abel function** for the group $\{g^\alpha\}$. For future reference, observe that if L is an Abel function for a group $\{g^\alpha\}$, then so is the function $L + c$ where c is an arbitrary real number. Also, note that when $\{g^\alpha\}$ is fixed point free, it it w.l.o.g. to assume that $g^\alpha(k) > k$ for all k in the domain of g^α and all $\alpha > 0$. Else, we can relabel the group by taking $\tilde{g}^\alpha := g^{-\alpha}$ for every $\alpha \in (-\lambda, \lambda)$. Under this assumption, any Abel function L for the iteration group is strictly increasing. Going back to our example, suppose D is an interval such that all functions g^α, $\alpha \neq 0$, are fixed point free when restricted to D. For instance, we can take $D = (k^*, +\infty)$. Then, $L(k) := \log_b(k - k^*)$, $k \in D$, is an Abel function for the group $\{g^\alpha|_D\}$ on D.

Given a topological space $\mathcal{Z}$ and a set $A \subset \mathcal{Z}$, we use A° to denote the topological interior of A. For a sequence $(A_n)_n$ of sets in $\mathcal{Z}$, we denote by $\mathrm{Ls}A_n \subset \mathcal{Z}$ and $\mathrm{Li}A_n \subset \mathcal{Z}$

the topological lim sup and lim inf of the sequence. See Aliprantis and Border [1, p.109] for precise definitions of these concepts. We write $A_n \to_L A$ if $A = \mathrm{Li} A_n = \mathrm{Ls} A_n$. The set A is called the **closed limit** of $(A_n)_n$. Following Lundberg [26], a correspondence $f^* : C \rightrightarrows \mathbb{R}$, where C is an interval in $\mathbb{R}$, is called a **cliff function** if the set $f^*(k)$ is connected for every $k \in C$. A cliff function f^* is **increasing** if $k \le k'$ and $l \in f^*(k), l' \in f^*(k')$ imply $l \le l'$ for all $k, k' \in C$. Observe that any increasing function $f : C \to \mathbb{R}$ is an increasing cliff function. If we identify every cliff function f^* with its graph in $\mathbb{R} \times \mathbb{R}$, we can also speak of the closed limit of a sequence $(f_n^*)_n$ of cliff functions.

A.1.4 Constructing an Iteration Group

Return to the proof of Theorem 1. Let $\succeq$ be a path stationary preference relation $\succeq$ with a representation (U, ϕ, I). Choose a sequence $(x_n)_n$ in X satisfying property (5) in Lemma 8. Let $C := \{U(z, d) : d \in X^\infty\}$ and note that C is a closed interval in $\mathbb{R}$ with nonempty interior. For every $k \in C$ and $n \in \mathbb{N}$, let $f_n(k) := \phi(x_n, k)$. Also, let $f(k) := \phi(z, k), k \in C$. By the choice of $(x_n)_n$, $f_n(C) \subset C$ for every n and $f(C) \subset C$.

Next, let $A \in \cup_t \mathcal{F}_t$ be an essential event. It is w.l.o.g. to assume that $A \in \mathcal{F}_1$. Letting $\mathcal{H}(A)$ be the subset of acts $h \in \mathcal{H}$ that are $\{A, A^c\}$-adapted, identify $U \circ \mathcal{H}(A) := \{U \circ h : h \in \mathcal{H}(A)\}$ with a subset in $\mathbb{R}^2$. Observe that $C^2 \subset U \circ \mathcal{H}(A)$. Thus, I induces a function on C^2. Abusing notation, use I to denote the induced function on C^2 as well and write $I(k, k')$ for its value at $(k, k') \in C^2$. By State Independence and Finite Continuity, I is a strictly increasing and continuous function on C^2.

The next lemma shows that the time aggregator ϕ is uniformly continuous in its first argument. The proof is technical, invoking several results from Lundberg [26] which we do not reproduce here. The reader can skip the proof without loss of continuity.

Lemma 9 $f_n \to_L f$.

Proof. Because $\succeq$ is continuous, $(f_n)_n$ converges pointwise to f. To prove the stronger form of convergence, identify each function f_n with an increasing cliff function. By Lundberg [26, Lemma 1.1], there is a subsequence of $(f_{n_m})_m$ and a cliff function f^* such that $f_{n_m} \to_L f^*$. We wish to show that $f^* = f$. In principle, f^* may be a correspondence rather than a proper function. The first step uses Path Stationarity to rule out this possibility. Namely, the fact that ϕ and I permute implies that

$$f_{n_m} I(k, k') = I(f_{n_m}(k), f_{n_m}(k')) \quad \forall m \in \mathbb{N}, \forall k, k' \in C. \tag{A.4}$$

By Lundberg [26, Lemma 4.8], we must then have

$$f^*(I(k,k')) = \{I(l,l') : l \in f^*(k), l' \in f^*(k')\} \quad \forall k,k' \in C. \tag{A.5}$$

By Lundberg [26, Lemma 4.7], f^* is a proper function. Next, from Lundberg [26, Lemma 1.2], we know that $(f_{n_m})_m$ converges to f^* uniformly on all compact subsets A of the interior of C. But then, $f^*(k) = f(k)$ for all $k \in C^\circ$. Since f, f^* are both continuous functions, we conclude that $f = f^*$. Since the convergent subsequence $(f_{n_m})_m$ was arbitrary, the original sequence $(f_n)_n$ has a closed limit given by f. ■

For every n, let $g_n := f^{-1} \circ f_n$. A direct calculation shows that the equality in (A.4) is preserved if the functions f_n are replaced with g_n. It follows from Lemma 9 and Lundberg [25, Thm 5.3] that $g_n \to_L j$. Letting $\operatorname{Dom} g_n$ denote the domain of g_n, it is also the case that $\operatorname{Dom} g_n \to_L C$. See Lundberg [25, Lemma 3.9]. Deduce from Lundberg [26, 4.16] that the sequence $(g_n)_n$ **generates an iteration group** $\{g^\alpha : \alpha \in (-\lambda, \lambda)\}$ over C°, that is, there is an iteration group $\{g^\alpha : \alpha \in (-\lambda, \lambda)\}$ such that for every $\alpha \in (-\lambda, \lambda)$ there is a sequence $(p_n)_n$ of integers such that $g_n^{p_n} \to_L g^\alpha$. Also,

$$g^\alpha I(k,k') = I(g^\alpha(k), g^\alpha(k')), \quad \forall \alpha \in (-\lambda, \lambda), \forall k,k' \in C^\circ. \tag{A.6}$$

The next lemma confirms that an analogue of (A.6) continues to hold when I is viewed as a function from $B^0_{U(x^\infty)}$, not as its restriction to C^2.

Lemma 10 *$I(g^\alpha \circ \xi) = g^\alpha I(\xi)$ for all $\alpha \in (-\lambda, \lambda)$ and $\xi \in B^0_{C^\circ}$.*

Proof. Because I and ϕ permute, we know that $I(g_n^m \circ \xi) = g_n^m I(\xi)$ for all $n \in \mathbb{N}, m \in \mathbb{Z}$, and $\xi \in B^0_{C^\circ}$. Fix $\xi \in B^0_{C^\circ}$ and $\alpha \in (-\lambda, \lambda)$, and let $\mathcal{F}'$ be a finite algebra on Ω such that ξ is $\mathcal{F}'$-measurable. Because the sequence $(g_n)_n$ generates the iteration group $\{g^\alpha\}$, for every α there is a sequence $(p_n)_n$ of integers such that g^α is the closed limit of the sequences $(g_n^{p_n})_n$. Moreover, $g^\alpha \circ \xi$ and each function $g_n^{p_n} \circ \xi$ are $\mathcal{F}'$-measurable. The desired equality follows since I is finite continuous and each function $g_n^{p_n}$ is continuous. ■

A.1.5 Constructing an Abel Function

The next step is to find a nonempty open interval $O \subset C^0$ on which the iteration group $\{g^\alpha : \alpha \in (-\lambda, \lambda)\}$ is fixed point free. The key is Theorem 4.13 in Lundberg [26] which shows that if an equation like (A.6) holds, then each $g^\alpha, \alpha \neq 0$, has at most one fixed point.

Lemma 11 *There exists a nonempty, open interval $O \subset C^\circ$ such that none of the functions $g^\alpha|_O, \alpha \neq 0$, has a fixed point.*

Proof. Let $C^\alpha \subset C$ be the domain of the function g^α. By Lundberg [26, Theorem 4.13], each function $g^\alpha, \alpha \neq 0$, has at most one fixed point. If none of the functions $g^\alpha, \alpha \neq 0$, has a fixed point, we can let $O = C^\circ$. Suppose instead that for some $\alpha^* \neq 0$, the function g^{α^*} has a fixed point $k^{\alpha^*} \in C^\circ$. Because C^{α^*} is an interval, there is $\varepsilon > 0$ such that either $(k^{\alpha^*} - \varepsilon, k^{\alpha^*}) \subset C^\alpha$ or $(k^{\alpha^*}, k^{\alpha^*} + \varepsilon) \subset C^\alpha$. Suppose the latter is true. An analogous argument applies to the other case. It is enough to show that there is $\varepsilon' \in (0, \varepsilon)$ such that no function $g^\alpha, \alpha \neq 0$, has a fixed point when restricted to the interval $(k^{\alpha^*}, k^{\alpha^*} + \varepsilon')$. If not, we can find a sequence $(\alpha_n)_n, \alpha_n \neq 0$, such that each function g^{α_n} has a fixed point k^{α_n} and $k^{\alpha_n} \searrow k^{\alpha^*}$. But then, $k^{\alpha_n} \in C^{\alpha^*}$ for all n large enough. From the properties of an iteration group, see Section A.1.3, we know that if some $g^\alpha, \alpha \neq 0$, has a fixed point k^α, then k^α is a fixed point of all other functions $g^{\alpha'}$ that are defined at k^α. Conclude that g^{α^*} has countably many fixed points, k^{α^*} and k^{α_n} for all n large enough, contradicting the fact that g^{α^*} has a unique fixed point. ∎

As observed in Section A.1.3, the iteration group $\{g^\alpha : \alpha \in (-\lambda, \lambda)\}$ over C° induces an iteration group $\{g^\alpha|_O : \alpha \in (-\hat{\lambda}, \hat{\lambda})\}$ over O. Since the latter group is fixed point free, it has an Abel function $L : O \to \mathbb{R}$. As was further explained in Section A.1.3, it is w.l.o.g. to assume that L is strictly increasing. We summarize these observations in the next lemma.

Lemma 12 *The iteration group $\{g^\alpha|_O : \alpha \in (-\hat{\lambda}, \hat{\lambda})\}$ has an Abel function $L : O \to \mathbb{R}$ which is strictly increasing.*

A.1.6 A Local Abel Function is Enough

The next lemma formalizes an important step discussed in Section 7.2. Namely, we plan to use the Abel function L to monotonically transform the representation (U, ϕ, I) into a new, more tractable representation. The problem is that L is defined only on a subinterval O of the range $U(X^\infty)$ of all possible utility levels. To address this problem, the next lemma shows that we can use Path Stationarity to "scale down" the representation (U, ϕ, I) into another representation whose utility levels are contained in O. Moreover, we can do so without changing the certainty equivalent I. The latter means that we preserve the connection between I and the iteration group $\{g^\alpha\}$, which was established in Lemma 10 and which is the key to the rest of the proof.

Lemma 13 *$\succeq$ has a representation $(\hat{U}, \hat{\phi}, \hat{I})$ such that $\hat{U}(X^\infty) =: D \subset O$ and $\hat{I}$ is equal to the restriction of I to B^0_D.*

Proof. Since $U : X^\infty \to \mathbb{R}$ is continuous and X is connected, we know that $\{U(x, x, ...) : x \in X\}$ is connected. It follows from property (3) in Lemma 8, that $\{U(x, x, ...) : x \in X\} = U(X^\infty)$. Thus, we can find $x_0 \in X$ such that $U(x_0, x_0, ...) \in O$. Let $f_0(k) := \phi(x_0, k)$ for all $k \in U(X^\infty)$. Because X is compact, there exists $N \in \mathbb{N}$ such that $f_0^N(U(X^\infty)) \subset O$. In addition, since f_0 is strictly increasing, f_0^N is strictly increasing. Thus, the function $\hat{U} := f_0^N \circ U : X^\infty \to \mathbb{R}$ represents the restriction of $\succeq$ to X^∞ and its range is contained in the set O. Moreover, if we let $\hat{\phi}(x, s) := f_0^N \phi(x, f_0^{-N}(s))$ for every $x \in X, s \in \hat{U}(X^\infty)$, then $\hat{\phi}$ is a time aggregator for $\hat{U}$. Let $\hat{I}$ be the restriction of I to B^0_D, where $D := \hat{U}(X^\infty) \subset O \subset C$. It remains to show that $(\hat{U}, \hat{\phi}, \hat{I})$ is a representation for $\succeq$. Let $x_0^N \in X^N$ be the N-dimensional vector each coordinate of which is equal to $x_0 \in X$. For all $h, h' \in \mathcal{H}$, Path Stationarity implies that

$$h \succeq h' \Leftrightarrow (x_0^N, h) \succeq (x_0^N, h') \Leftrightarrow I(f_0^N \circ U \circ h) \geq I(f_0^N \circ U \circ h') \Leftrightarrow I(\hat{U} \circ h) \geq I(\hat{U} \circ h'),$$

completing the proof. ∎.

A.1.7 A Monotone Transformation of Utility

It was observed in Section A.1.3 that if L is an Abel function for some iteration group, then so is the function $L + c$ where c is an arbitrary real number. Hence, we can assume that $0 \in L(D)^\circ$. Note that $L(D)^\circ$ is nonempty since $D^\circ = [\hat{U}(X^\infty)]^\circ$ is nonempty and L is strictly increasing. Now, use L to construct a monotone transformation of the representation $(\hat{U}, \hat{\phi}, \hat{I})$:

$$\begin{aligned} &\tilde{U} := L \circ \hat{U}, \\ &\tilde{\phi}(x, s) := L\hat{\phi}(x, L^{-1}(s)) \quad \forall x \in X, s \in L(D), \\ &\tilde{I}(\xi) := LI(L^{-1} \circ \xi) \quad \forall \xi \in B^0_{L(D)}. \end{aligned}$$

By construction, $(\tilde{U}, \tilde{\phi}, \tilde{I})$ is a representation of $\succeq$. The next lemma shows that the certainty equivalent $\tilde{I} : B^0_{L(D)} \to \mathbb{R}$ is translation invariant.

Lemma 14 *$\tilde{I}(\xi + \alpha) = \tilde{I}(\xi) + \alpha$ for all $\xi \in B^0_{L(D)}, \alpha \in (-\hat{\lambda}, \hat{\lambda})$ such that $\xi + \alpha \in B^0_{L(D)}$.*

Proof. Take ξ and α as in the statement of the lemma. Let $\xi' \in B^0_D$ be such that $L \circ \xi' = \xi$. Then,

$$\begin{aligned}\tilde{I}(\xi + \alpha) &= LI\big[L^{-1} \circ (L \circ \xi' + \alpha)\big] = LI[g^\alpha \circ \xi'] = Lg^\alpha I(\xi') = L(I(\xi')) + \alpha \\ &= LI[L^{-1} \circ \xi] + \alpha = \tilde{I}(\xi) + \alpha.\end{aligned}$$

The first equality follows from the definitions of $\tilde{I}$ and ξ', and the fact that $\hat{I}$ is the restriction of I to B^0_D. The second equality follows since L is an Abel function for the group $\{g^\alpha|_O : \alpha \in (-\hat{\lambda}, \hat{\lambda})\}$ and $D \subset O$. The third equality follows because, by Lemma 10, I and g^α permute. The fourth equality uses again the fact that L is an Abel function. The final two equalities follow from the definitions of ξ' and $\tilde{I}$ respectively.[20] ■

A.1.8 A Functional Equation

In this section, we use the translation invariance of $\tilde{I}$ to deduce a functional equation which is solved in Lundberg [27]. Once again, think of I as a function on $C^2 \subset \mathbb{R}^2$. Analogously, $\tilde{I}$ becomes a function on $L(D)^2$. Write $[c, c']$ for the closed interval $L(D)$ and define

$$\psi(k) := \begin{cases} \tilde{I}(c, c+k) - c & \text{if } k \in [0, c'-c] \\ \tilde{I}(c', c'+k) - c' & \text{if } k \in [c-c', 0] \end{cases}$$

Lemma 15 *The function ψ is continuous and strictly increasing; $\psi(0) = 0$; $\psi(k) < k$ for all $k > 0$, while $\psi(k) > k$ for all $k < 0$. Finally, the function $k \mapsto \psi(k) - k$ is strictly decreasing.*

Proof. ψ is continuous and strictly increasing since $\tilde{I}$ is continuous and strictly increasing. Since $\tilde{I}(k, k) = k$ for all $k \in [c, c']$, $\psi(0) = 0$. To see that $k \mapsto \psi(k) - k$ is a strictly decreasing function, pick $k \in [0, c'-c)$ and $\varepsilon > 0$ such that $k + \varepsilon \in [0, c'-c]$. Then,

$$\psi(k+\epsilon) - \psi(k) - \varepsilon = \tilde{I}(c, c+k+\varepsilon) - \tilde{I}(c+\varepsilon, c+k+\varepsilon) < 0.$$

[20] Arguments similar to the ones we use in this section and until the rest of the proof of Theorem 1 appear in our companion paper, Bommier et al. [4], as well. In their totality however, the proofs of Theorem 1 and of the corresponding result in Bommier et al. [4] contain many non-trivial differences. To save the reader the trouble of cross-checking two long and complicated proofs, we have written two separate, self-contained proofs.

Conclude that $\psi(k) - k$ is strictly decreasing on $[0, c' - c]$. A similar argument for $k \in [c - c', 0]$ shows that $\psi(k) - k$ is strictly decreasing on its entire domain. ■

Since, by Lemma 14, $\tilde{I}$ is translation invariant,

$$\tilde{I}(s,t) = s + \psi(t-s) \quad \forall s,t \in [c,c']. \tag{A.7}$$

For every $x \in X$, write $\tilde{f}_x$ for the function $\tilde{\phi}(x,\cdot)$ on $[c,c']$. Using (A.7), the fact that $\tilde{I}$ and $\tilde{\phi}$ permute implies that

$$\tilde{f}_x(s + \psi(t-s)) = \tilde{f}_x(s) + \psi(\tilde{f}_x(t) - \tilde{f}_x(s)) \quad \forall x \in X, s,t \in [c,c']. \tag{A.8}$$

The above is a special case of the functional equation

$$g(s + \psi(t-s)) = g(s) + \psi(g(t) - g(s)) \quad \forall s,t \in [c,c'], \tag{A.9}$$

which is solved in Lundberg [27] when the function g is continuous and strictly increasing and ψ has the properties listed in Lemma 15. Thinking of ψ as a known function and of g as a solution to (A.9), we break the remainder of the proof in two cases, depending on whether there is a solution g that is affine on some subinterval of its domain.

A.1.9 The Affine Case

Suppose that (A.9) is satisfied for some g that is affine on a subinterval of $[c,c']$. It follows from Lundberg [27, Thm 10.1,10.3] that all solutions g of (A.9) are affine on $[c,c']$. In particular,

$$\tilde{f}_x(k) = u(x) + b(x)k \quad \forall x \in X, k \in [c,c']. \tag{A.10}$$

Because $(\tilde{U}, \tilde{\phi}, \tilde{I})$ is a representation for $\succeq$ and $\succeq$ is continuous on X^∞, it follows that $u, b : X \to \mathbb{R}$ are continuous functions. Since each function $\tilde{f}_x$ is strictly increasing, we know that $b(x) > 0$. In addition, property (2) in Lemma 8 implies that $b(x) < 1$ for all $x \in X$. Thus, $\tilde{U} : X^\infty \to \mathbb{R}$ is an Uzawa-Epstein utility.

Turn to the certainty equivalent $\tilde{I} : B^0_{L(D)} \to \mathbb{R}$. By Lemma 14, we know that $\tilde{I}$ is translation invariant. The next lemma shows that $\tilde{I}$ is $b(x)$-homogeneous for all $x \in X$. To state the lemma, note that if $\xi \in B^0_{L(D)}$, then $b(x)\xi \in B^0_{L(D)}$. This is because $0 \in L(D)^\circ$ and $b(x) \in (0,1)$.

Lemma 16 *$\tilde{I}(b(x)\xi) = b(x)\tilde{I}(\xi)$ for all $x \in X, \xi \in B^0_{L(D)}$.*

Proof. Since $0 \in L(D)^\circ$, we can find $x_0 \in X$ such that $u(x_0) = 0$. Let $\beta := b(x_0) \in (0,1)$. The fact that $\tilde{I}$ and $\tilde{\phi}$ permute implies that

$$\tilde{I}\big[u(x) + b(x)\xi\big] = u(x) + b(x)\tilde{I}(\xi) \quad \forall x \in X, \xi \in B^0_{L(D)}. \tag{A.11}$$

Letting $x = x_0$, we obtain $\tilde{I}(\beta\xi) = \beta\tilde{I}(\xi)$ for all $\xi \in B^0_{L(D)}$. In turn, $\tilde{I}(\beta^t\xi) = \beta^t\tilde{I}(\xi)$ for all $\xi \in B^0_{L(D)}, t \in \mathbb{N}$.

Next, fix $x \in X$ and $\xi \in B^0_{L(D)}$. Choose t large enough so that $\beta^t u(x) \in (-\hat{\lambda}, \hat{\lambda})$. We claim that

$$\begin{aligned}\beta^t u(x) + \beta^t b(x)\tilde{I}(\xi) &= \tilde{I}\big[\beta^t u(x) + \beta^t b(x)\xi\big] = \beta^t u(x) + \tilde{I}\big[\beta^t b(x)\xi\big] \\ &= \beta^t u(x) + \beta^t\tilde{I}[b(x)\xi].\end{aligned}$$

The first equality follows from (A.11); the second because $\tilde{I}$ is translation invariant; the final equality follows because, as we showed earlier in this proof, $\tilde{I}$ is β-homogeneous. ■

Lemma 17 *$\tilde{I} : B^0_{L(D)} \to \mathbb{R}$ can be extended to a certainty equivalent $\tilde{I}^e : B^0 \to \mathbb{R}$ which is translation invariant and $b(x)$-homogeneous for all $x \in X$.*

Proof. For every $\xi \in B^0$, pick $t \in \mathbb{N}$ large enough so that $\beta^t\xi \in B^0_{L(D)}$ and let $\tilde{I}^e(\xi) := \beta^{-t}\tilde{I}(\beta^t\xi)$. One can verify, see Kochov [19, Lemma 11], that $\tilde{I}^e$ is well defined and extends $\tilde{I}$. To show that $\tilde{I}^e$ is translation invariant, take any $\xi \in B^0$ and any $\alpha \in \mathbb{R}$. Choose t large enough so that $\beta^t\xi, \beta^t(\xi + \alpha) \in B^0_{L(D)}$ and $\beta^t\alpha \in (-\hat{\lambda}, \hat{\lambda})$. Then,

$$\tilde{I}^e(\xi + \alpha) = \beta^{-t}\tilde{I}(\beta^t\xi + \beta^t\alpha) = \beta^{-t}(\tilde{I}(\beta^t\xi) + \beta^t\alpha) = \tilde{I}^e(\xi) + \alpha.$$

Similar arguments show that the extension $\tilde{I}^e$ is $b(x)$-homogeneous for every $x \in X$. By construction, $\tilde{I}^e$ is increasing and normalized. Finally, because $\tilde{I}^e$ is translation invariant, it is norm continuous. Thus, $\tilde{I}^e$ is a certainty equivalent in the sense of Section 2. ■

The next lemma shows that if $b : X \to (0,1)$ is nonconstant, then $\tilde{I}^e$ is positively homogeneous.

Lemma 18 *If a function $J : B^0 \to \mathbb{R}$ is γ-homogeneous for all γ in some nonempty open interval $(a,b) \subset (0,1)$, then J is positively homogeneous.*

Proof. It is clear that J is γ-homogeneous for all $\gamma \in (a^t, b^t)$ and all $t \in T$. Observe that $\log_b a > 1$ and pick k such that $1+\frac{1}{k} < \log_b a$. Then, $b^{t+1} > a^t$ for all $t \geq k$. Conclude that $(0, b^k) \subset \cup_t (a^t, b^t)$ and, hence, that J is γ-homogeneous for all $\gamma \in (0, b^k)$. Next pick any $\gamma > 0$ and $\xi \in B^0$. Choose $\beta \in (0, b^k)$ and t large enough so that $\beta^t \gamma \in (0, b^k)$. Because $\beta^t \in (0, b^k)$, J is β^t-homogeneous. Hence, $J(\beta^t \gamma \xi) = \beta^t J(\gamma \xi)$. Because J is $\beta^t \gamma$-homogeneous, $J(\beta^t \gamma \xi) = \beta^t \gamma J(\xi)$. The last two equalities prove that J is γ-homogeneous. ■

A.1.10 The Non-Affine Case

Suppose now that (A.9) has no solution g that is affine on a subinterval of $[c, c']$. It follows from Lundberg [27, Thm. 11.1] that all solutions g and, in particular, all functions $\tilde{f}_x$ take the form

$$\tilde{f}_x(k) = \tilde{\phi}(x,k) = \frac{1}{p} \ln(u(x) + b(x) e^{pk}) \quad \forall x \in X, k \in [c, c'], \tag{A.12}$$

where $u, b : X \to \mathbb{R}$, $p \in \mathbb{R}$ and $p \neq 0$. Assume that $p > 0$ and let $H(s) := e^{ps}$ for every $s \in \mathbb{R}$. If $p < 0$, we can let $H(s) := -e^{ps}$ and the subsequent analysis would carry through in an analogous manner. Let $D^* := [e^{pc}, e^{pc'}]$ and define $U^* := H \circ \tilde{U}$, $\phi^*(x,k) := H\tilde{\phi}(x, H^{-1}(k))$, and $I^*(\xi) := H\tilde{I}(H^{-1} \circ \xi)$ for all $x \in X, k \in D^*, \xi \in B^0_{D^*}$. Then,

$$\phi^*(x,k) = u(x) + b(x)k, \quad \forall x \in X, k \in D^*. \tag{A.13}$$

By construction, (U^*, ϕ^*, I^*) is a representation for $\succeq$. Once again, the functions $u, b : X \to \mathbb{R}$ are continuous and $b(x) \in (0,1)$ for every $x \in X$. Thus, U^* is an Uzawa-Epstein utility. Turn to the certainty equivalent $I^* : B^0_{D^*} \to \mathbb{R}$ and note that the open interval $(H(-\hat{\lambda}), H(\hat{\lambda}))$ contains 1.

Lemma 19 *$I^*(\gamma\xi) = \gamma I^*(\xi)$ for all $\gamma \in (H(-\hat{\lambda}), H(\hat{\lambda}))$ and $\xi \in B^0_{D^*}$ such that $\gamma\xi \in B^0_{D^*}$.*

Proof. Let ξ and γ be as in the statement of the lemma. Let $\xi' := H^{-1} \circ \xi$ and $\alpha := H^{-1}(\gamma)$. Observe that $H^{-1} \circ (\gamma\xi) = \xi' + \alpha$. Also, $\gamma' + \alpha \in B^0_{L(D)}$ and $\alpha \in (-\hat{\lambda}, \hat{\lambda})$. we claim that

$$I^*(\gamma\xi) = H\tilde{I}\big[H^{-1} \circ (\gamma\xi)\big] = H\big[\tilde{I}(\xi' + \alpha)\big] = H\big[\tilde{I}(\xi') + \alpha\big] = H\big[\tilde{I}(\xi')\big]H(\alpha) = I^*(\xi)\gamma.$$

The first equality follows from the definition of I^*, the second from the definition of ξ', the third from the translation invariance of $\tilde{I}$ (Lemma 14), the fourth equality from the fact that H is an exponential function, and the final equality from the definition of I^*. ■

Because γ is restricted to lie in the interval $(H(-\hat{\lambda}), H(\hat{\lambda}))$, one can think of the property established in Lemma 19 as a type of "local" homogeneity. The next lemma shows that I^* satisfies a similarly "local" type of translation invariance.

Lemma 20 *For every ξ in the interior of $B^0_{D^*}$, there is $k_\xi > 0$ such that $I^*(\xi + k) = I^*(\xi) + k$ for all k such that $|k| \leq k_\xi$.*

Proof. Because I^* and ϕ^* permute,

$$u(x) + b(x)I^*(\xi) = I^*(u(x) + b(x)\xi) \quad \forall x \in X, \xi \in B^0_{D^*}. \tag{A.14}$$

Fix some ξ in the interior of $B^0_{D^*}$. For every $x, x' \in X$, define

$$\xi' = \frac{u(x') - u(x)}{b(x)} + \frac{b(x')}{b(x)}\xi.$$

Note that if x' is sufficiently close to x, then $\xi' \in B^0_{D^*}$. Using (A.14), deduce that

$$I^*(\xi) = \frac{u(x) - u(x')}{b(x')} + \frac{b(x)}{b(x')}I^*\Big(\frac{u(x') - u(x)}{b(x)} + \frac{b(x')}{b(x)}\xi\Big).$$

Moreover, if x' is sufficiently close to x, we can apply Lemma 19 and deduce that

$$I^*(\xi) = \frac{u(x) - u(x')}{b(x')} + I^*\Big(\frac{u(x') - u(x)}{b(x')} + \xi\Big).$$

If $u : X \to \mathbb{R}$ is a nonconstant function, then the above equality completes the proof. Suppose then that $u : X \to \mathbb{R}$ is constant. Because $U^* : X^\infty \to \mathbb{R}$ cannot be constant, it follows that $b : X \to (0,1)$ cannot be constant either. Moreover, from (A.14) we can deduce that

$$u(x) + b(x)u(x) + b^2(x)I^*(\xi) = I^*\Big(u(x) + b(x)u(x) + b^2(x)\xi\Big) \quad \forall x \in X, \xi \in B^0_{D^*}.$$

Letting $v(x) := u(x) + b(x)u(x)$ and $c(x) := b^2(x)$, we see that

$$v(x) + c(x)I^*(\xi) = I^*(v(x) + c(x)\xi) \quad \forall x \in X, \xi \in B^0_{D^*}.$$

Since by construction v is not constant, the proof reduces to the case when u is not constant. ■

The next lemma shows that the local type of translation invariance established in Lemma 20 integrates into a global property.

Lemma 21 *$I^*(\xi + k) = I^*(\xi) + k$ for all $\xi \in B^0_{D^*}$ and $k \in \mathbb{R}$ such that $\xi + k \in B^0_{D^*}$.*

Proof. Fix some ξ in the interior of $B^0_{D^*}$ and some $k > 0$ such that $\xi + k \in B^0_{D^*}$. Analogous arguments apply when $k < 0$. Let $k^* \geq 0$ be the largest k' such that $I^*(\xi + k'') = I^*(\xi) + k''$ for all $k'' \in [0, k']$. By Lemma 20, $k^* > 0$. If $k \leq k^*$, we are done. Suppose $k > k^*$ and, by way of contradiction, that $I^*(\xi + k) \neq I^*(\xi) + k$. Because $k > k^*$, $\xi' := \xi + k^*$ is in the interior of $B^0_{D^*}$. By Lemma 20, there is $k^{**} > 0$ such that $I^*(\xi' + k') = I^*(\xi') + k'$ for all $k' \in [0, k^{**}]$. But then, for all such k',

$$I^*(\xi + k^* + k') = I^*(\xi + k^*) + k' = I^*(\xi) + k^* + k',$$

contradicting the definition of k^*. ■

Similarly, the next lemma shows that the local type of homogeneity established in Lemma 19 integrates into a global property. The proof, which is analogous to that of Lemma 21, is omitted.

Lemma 22 *$I^*(\gamma\xi) = \gamma I^*(\xi)$ for all $\gamma \geq 0$ and $\xi \in B^0_{D^*}$ such that $\gamma\xi \in B^0_{D^*}$.*

We need one more property of I^*.

Lemma 23 *$I^*(\alpha\xi + (1-\alpha)k) = \alpha I^*(\xi) + (1-\alpha)k$ for all $\alpha \in [0,1], \xi \in B^0_{D^*}, k \in D^*$.*

Proof. Suppose ξ is in the interior of $B^0_{D^*}$. For all $\alpha \in [0,1]$ sufficiently close to 1, $\alpha\xi \in B^0_{D^*}$. By the translation invariance of I^* (Lemma 20) and the positive homogeneity of I^* (Lemma 22),

$$I^*(\alpha\xi + (1-\alpha)k) = I^*(\alpha\xi) + (1-\alpha)k = \alpha I(\xi) + (1-\alpha)k.$$

Thus, we have once again established a local version of the property we want to prove. Arguments analogous to those in Lemma 21 show that the desired property holds globally, that is, for all $\alpha \in [0,1]$. ■

Lemma 24 *$I^* : B^0_{D^*} \to \mathbb{R}$ can be uniquely extended to a translation invariant and positively homogeneous certainty equivalent $I^{*e} : B^0 \to \mathbb{R}$.*

Proof. Fix some k in the interior of D^*. For every $\xi \in B^0$, there exists $\alpha \in (0,1)$ such that $\alpha\xi + (1-\alpha)k \in B^0_{D^*}$. Let

$$I^{*e}(\xi) := \frac{1}{\alpha} I^*\big(\alpha\xi + (1-\alpha)k\big) - \frac{1-\alpha}{\alpha}k.$$

To see that I^{*e} is well defined, take $\alpha_1 > \alpha_2$ such that

$$\xi_1 := \alpha_1\xi + (1-\alpha_1)k \in B^0_{D^*} \quad \text{and} \quad \xi_2 := \alpha_2\xi + (1-\alpha_2)k \in B^0_{D^*}.$$

By construction, $\xi_2 = \frac{\alpha_2}{\alpha_1}\xi_1 + (1 - \frac{\alpha_2}{\alpha_1})k$. By Lemma 23, $I^*(\xi_2) = \frac{\alpha_2}{\alpha_1}I(\xi_1) + (1 - \frac{\alpha_2}{\alpha_1})k$, which is equivalent to

$$\frac{1}{\alpha_1}I^*(\xi_1) - \frac{1-\alpha_1}{\alpha_1}k = \frac{1}{\alpha_2}I^*(\xi_2) - \frac{1-\alpha_2}{\alpha_2}k.$$

Thus, I^{*e} is well defined. Messy but straight-forward calculations show that I^{*e} is translation invariant. Using translation invariance and Lemma 23, one can then show that I^{*e} is positively homogeneous. By construction, I^{*e} is increasing and normalized. Finally, I^{*e} is norm continuous because it is increasing and translation invariant. Thus, I^{*e} is a certainty equivalent in the sense of Section 2. ■

A.2 Proof of Theorem 3

Let $f : U(X^\infty) \to \hat{U}(X^\infty)$ by such that $f(U(d)) = \hat{U}(d)$ for all $d \in X^\infty$. Because $U, \hat{U} : X^\infty \to \mathbb{R}$ represent the same preference relation on X^∞, f is well defined and strictly increasing. Because $\hat{U}$ is continuous and X^∞ is connected, the set $\hat{U}(X^\infty) = f(U(X^\infty)) \subset \mathbb{R}$ is connected. Thus, f is continuous. Let $z \in X$ be such that $(z, z, ...) \in X^\infty$ attains the minimum of $U : X^\infty \to \mathbb{R}$. Such a $z \in X$ exists by Lemma 10. Renormalizing if necessary, assume that $u(z) = 0$. Let E be the set of all points in the interior of $U(X^\infty)$ at which f is differentiable. Because f is increasing, E^c has outer measure zero. See Royden [29, Thm 3, p.100]. Moreover, for every $x \in X$, $s \in U(X^\infty)^\circ$, and $k \in \mathbb{R}$ small enough, we have

$$\frac{f\big[u(x) + b(x)(s+k)\big] - f\big[u(x) + b(x)s\big]}{b(x)k} = \frac{\hat{b}(x)}{b(x)} \frac{f(s+k) - f(s)}{k}. \tag{A.15}$$

Thus, if f is differentiable at $s \in U(X^\infty)^\circ$, then f is differentiable at every point in the set

$$A(s) \coloneqq \{u(x) + b(x)s : x \in X\}.$$

We want to show that f is differentiable at each $s \in U(X^\infty)^\circ$. Each set $A(s), s \in E$, is an interval because X is connected and $u, b : X \to \mathbb{R}$ are continuous. Also, $s \in A(s)$ for every $s \in E$. Because E^c has zero measure, for every s' in the interior of $U(X^\infty)$, there is $s \in E$ such that $s > s' > 0$. Thus, it suffices to show that f is differentiable on $(0, s]$ for every $s \in E$. Fix such an s and let $s_n \coloneqq b(z)^n s, n \in \mathbb{N}$. Since $u(z) = 0$, $s_n \in A(s)$ for every n. Since $s \in E$, f is differentiable at every s_n. Also,

$$s_{n+1} \in A(s_{n+1}) \cap A(s_n) \quad \forall n. \tag{A.16}$$

Conclude that the set $\cup_n A(s_n)$ is connected. Since $s_n \to_n 0$, it follows that $(0, s] \subset \cup_n A(s_n)$. Since f is differentiable on every $A(s_n)$, we are done.

Next, we deduce from (A.15) that

$$b(x) f'\big[u(x) + b(x)s\big] = \hat{b}(x) f'(s) \quad \forall x \in X, s \in U(X^\infty)^\circ. \tag{A.17}$$

We want to show that the derivative f' is never 0. By way of contradiction, suppose $f'(s) = 0$ for some $s \in U(X^\infty)^\circ$. Since $b(x), \hat{b}(x) > 0$ for all $x \in X$, it follows from (A.17) that

$$f'\big[u(x) + b(x)s\big] = 0 \quad \forall x \in X.$$

Conclude that $f'(s') = 0$ for all $s' \in A(s)$. This contradicts the fact that f is strictly increasing.

The next step is to show that $b = \hat{b}$. Take some $x \in X$ such that $s := u(x)(1 - b(x))^{-1} \in U(X^\infty)^\circ$. Plugging x and s into (A.17) gives

$$b(x) f'(s) = \hat{b}(x) f'(s).$$

Since $f'(s) \neq 0$, we are done.

The final step is to show that f is affine. Since $b = \hat{b}$, (A.17) becomes

$$f'\big[u(x) + b(x)s\big] = f'(s) \quad \forall x \in X, \forall s \in U(X^\infty). \tag{A.18}$$

Fix some $s \in U(X^\infty)^\circ$ and once again let $s_n := b(z)^n s$. It follows from (A.18) that f' is constant on each interval $A(s_n)$. As argued above, $A(s_n) \cap A(s_{n+1}) \neq \varnothing$ for every n. Also, $(0, s] \subset \cup_n A(s_n)$. In particular, f' is constant on $(0, s]$ for every $s \in U(X^\infty)^\circ$. Conclude that f is affine on $U(X^\infty)^\circ$ and, since f is continuous, that f is affine.

A.3 Proof of Theorem 2

Let $\mathcal{H}^{rp}$ be the set of all repeating permutation acts $h \in \mathcal{H}$.

Lemma 25 *$U \circ \mathcal{H}^{rp}$ is dense in $U \circ \mathcal{H}$.*

Proof. Take some act $h \in \mathcal{H}$. Since h is finite, there is some $t \in T$ and a finite partition $\Pi := \{A_1, ..., A_n\} \subset \mathcal{F}_t$ such that h is Π-adapted. Fix some ω_i from each set A_i and write $(x_0^i, x_1^i, ...)$ for $h(\omega_i) \in X^\infty$. For every $k \in T$ and every i, let $a_i^k := (x_0^i, ..., x_k^i)$. Let $b_1^k := (a_1^k, a_2^k, ..., a_n^k)$, $b_2^k := (a_2^k, a_3^k, ..., a_1^k)$,..., and $b_n^k := (a_n^k, a_1^k, ..., a_{n-1}^k)$. Let h^k be the Π-adapted act such that $h^k(\omega) = (b_i^k, b_i^k, ...)$ for all $\omega \in A_i$ and i. Observe that $h^k \in \mathcal{H}^{rp}$ and $h^k \to_k h$. ■

To develop intuition about the proof of Theorem 2, it is helpful to restate IHUT in terms of a utility representation (u, b, I). First, we need some notation. Given $t \in T$, write a for a list $(x_0, x_1, ..., x_{t-1}) \in X^t$ of outcomes as well as for a list $(f_0, ..., f_{t-1})$ of functions from Ω into X. Then, every repeating permutation act $h \in \mathcal{H}^{rp}$ can be written as a sequence $(a, a, ...)$ for some list $a = (f_0, f_1, ..., f_{t-1})$ of functions and some $t \in T$. If $h = (a, a, ...)$ is a repeating permutation act, we also know that for every $\omega, \omega' \in \Omega$, the lists

$$(f_0(\omega), f_1(\omega), ..., f_{t-1}(\omega)) \in X^t \quad \text{and} \quad (f_0(\omega'), f_1(\omega'), ..., f_{t-1}(\omega')) \in X^t$$

are permutations of one another. Given a function $b : X \to (0, 1)$, it follows that

$$\Pi_{k=0}^{t-1} b(f_k(\omega)) = \Pi_{k=0}^{t-1} b(f_k(\omega')) \quad \forall \omega, \omega' \in \Omega.$$

Going back to the repeating permutation act $h = (a, a, ...)$, we can thus let

$$b(a) := \Pi_{k=0}^{t-1} b(f_k(\omega)) \tag{A.19}$$

and be certain that $b(a)$ is a number in $(0, 1)$ independent of $\omega \in \Omega$.

Next, suppose $\succeq$ is a path stationary preference relation on $\mathcal{H}$ with a representation (u, b, I). Let $h, g, m \in \mathcal{H}$ be as in the statement of IHUT. Since h is a repeating permutation act, $h = (a, a, ...)$ for some list a of functions. Defining $b(a)$ as in (A.19), observe that

$$U \circ m = (1 - b(a))[U \circ h] + b(a)[U \circ g].$$

Moreover, IHUT becomes equivalent to the implication:

$$I[U \circ h] \geq I[U \circ g] \Rightarrow I\Big((1 - b(a))[U \circ h] + b(a)[U \circ g]\Big) \geq I[U \circ g]. \tag{A.20}$$

If I is quasiconcave, then (A.20) is true by definition. On its own, (A.20) is strictly weaker than quasiconcavity since $U \circ h$ is restricted to lie in the set $U \circ \mathcal{H}^{rp}$ and since the mixing weight $b(a)$ is a function of h. Our strategy is to show that the set $U \circ \mathcal{H}^{rp}$ is sufficiently rich for (A.20) to imply quasiconcavity. From Lemma 25, we already know that $U \circ \mathcal{H}^{rp}$ is dense in $U \circ \mathcal{H}$. Roughly, the bulk of the remaining proof is to show that $U \circ \mathcal{H}^{rp}$ (or some transformation thereof) contains an open set O. Then, (A.20) implies that I is quasiconcave within O and, since I is homogeneous, that I is quasiconcave.

To state the next lemma, fix a repeating permutation act $(a, a, ...) \in \mathcal{H}^{rp}$ and some $x \in X$.

Lemma 26 *If* $(a,a,...)\sim(x,x,...)$, *then* $(a,x,a,x,...)\sim(x,x,...)$.

Proof. Let $h := (a,x,a,x,...)$. First, we are going to show that $h \succsim (x,x,...)$. By PS, $(x,a,a,a,...)\sim(x,x,...)\sim(a,a,...)$. By IHUT, $(a,x,a,a,a,...)\succsim(a,a,...)$. By PS again,

$$(x,a,x,a,a,a,...)\succsim(x,a,a,...)\sim(a,a,...).$$

By IHUT again, $(a,x,a,x,a,a,a,...)\succsim(a,a,...)$. Repeating the argument and using the fact that $\succsim$ is continuous shows that $h\succsim(x,x,...)$. By way of contradiction, suppose now that $h\succ(x,x,...)$. Then, $h\succ(x,h)\succ(x,x,...)$. But $(x,h)=(x,a,x,a,...)$, that is, (x,h) is a repeating permutation act. By IHUT, we conclude that $(x,a,h)\succsim(x,h)$. By PS, $(a,h)\succsim h$ and, hence, $(a,h)\succ(x,x,...)$. The latter implies that $(a,h)\succ(x,a,h)$. To summarize, we have

$$(a,h)\succ(x,a,h)\succsim(x,h)=(x,a,x,a,...).$$

By IHUT again, $(x,a,a,h)\succsim(x,h)$. By PS, $(a,a,h)\succsim h$. Similarly, $(a,a,a,h)\succsim h$. Iterating the argument and using the fact that $\succsim$ is continuous, we deduce that $(a,a,...)\succsim h$. Altogether, we have

$$(a,a,...)\succsim h\succ(x,x,...)\sim(a,a,...),$$

a contradiction. ■

Next, let $\Pi\subset\cup_t\mathcal{F}_t$ be a finite partition of Ω. To complete the proof of Theorem 2, it is enough to show that I is positively homogeneous and quasiconcave when I is restricted to the space of Π-measurable functions in B^0. Based on PS, it is w.l.o.g. to assume that $\Pi\subset\mathcal{F}_1$. To simplify the exposition, we also assume that Π contains three sets so that $\Pi=\{A_1,A_2,A_3\}$. The arguments extend naturally to all finite partitions Π.

Assume that utility is normalized so that $0\in u(X)^\circ$. Let $x_0\in X$ be such that $u(x_0)=0$ and let $\beta:=b(x_0)$. Fix some $(y_0,y_1,y_2)\in X^3$ and consider the Π-measurable functions $f_0,f_1,...,f_{11}$ from Ω into X defined as

	A_1	A_2	A_3		A_1	A_2	A_3		A_1	A_2	A_3
$f_3:$	x_0	x_0	y_0	$f_7:$	y_1	x_0	x_0	$f_{11}:$	x_0	y_2	x_0
$f_2:$	x_0	y_0	x_0	$f_6:$	x_0	x_0	y_1	$f_{10}:$	y_2	x_0	x_0
$f_1:$	y_0	x_0	x_0	$f_5:$	x_0	y_1	x_0	$f_9:$	x_0	x_0	y_2
$f_0:$	x_0	x_0	x_0	$f_4:$	x_0	x_0	x_0	$f_8:$	x_0	x_0	x_0

Let

$$\begin{aligned} a &:= (f_0, f_1, ..., f_{11}) \\ h &:= (a, a, ...) \\ h' &:= (a, x_0, x_0, ...) \\ \hat{b}(y_0, y_1, y_2) &:= b(y_0)b(y_1)b(y_2)\beta^9. \end{aligned}$$

Thus defined, h is a repeating permutation act and

$$U \circ h = \frac{1}{1 - \hat{b}(y_0, y_1, y_2)} U \circ h'.$$

As we vary the choice of $(y_0, y_1, y_2) \in X^3$, we obtain different acts h and h'. Let $\mathcal{H}_3^{rp}$ be the space of all repeating permutation acts h obtained in this manner. Also, let Φ be the function

$$(y_0, y_1, y_2) \mapsto U \circ h'.$$

Since each function $U \circ h' : \Omega \to \mathbb{R}$ is Π-measurable, we can identify $U \circ h'$ with a vector in $\mathbb{R}^3$. Then, Φ becomes a function from X^3 into $\mathbb{R}^3$.

Lemma 27 $\Phi(X^3)$ *has nonempty interior in* $\mathbb{R}^3$*. In particular,* $\mathbf{0} \in \Phi(X^3)^\circ$.

Proof. By construction,

$$\beta^{-1}\Phi(y_0, y_1, y_2) = \begin{bmatrix} 1 & \beta^5 b(y_0) & \beta^7 b(y_0)b(y_1) \\ \beta & \beta^3 b(y_0) & \beta^8 b(y_0)b(y_1) \\ \beta^2 & \beta^4 b(y_0) & \beta^6 b(y_0)b(y_1) \end{bmatrix} \begin{bmatrix} u(y_0) \\ u(y_1) \\ u(y_2) \end{bmatrix}.$$

If we let $v_0 := u(y_0)$, $v_1 := b(y_0)u(y_1)$ and $v_2 := b(y_0)b(y_1)u(y_2)$, we can rewrite the above expression as

$$\beta^{-1}\Phi(y_0, y_1, y_2) = \begin{bmatrix} 1 & \beta^5 & \beta^7 \\ \beta & \beta^3 & \beta^8 \\ \beta^2 & \beta^4 & \beta^6 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix}$$

The 3×3 matrix in the last expression has full rank. Hence, the linear mapping

$$L(\tilde{v}_0, \tilde{v}_1, \tilde{v}_2) := \begin{bmatrix} 1 & \beta^5 & \beta^7 \\ \beta & \beta^3 & \beta^8 \\ \beta^2 & \beta^4 & \beta^6 \end{bmatrix} \begin{bmatrix} \tilde{v}_0 \\ \tilde{v}_1 \\ \tilde{v}_2 \end{bmatrix}$$

from $\mathbb{R}^3$ into $\mathbb{R}^3$ maps open sets into open sets. Now, consider the set

$$C := \left\{ (u(y_0), b(y_0)u(y_1), b(y_0)b(y_1)u(y_2)) : (y_0, y_1, y_2) \in X^3 \right\}$$

Let $\underline{b} := \min_{x \in X} b(X) \in (0,1)$. The latter is well defined since the function $b : X \to (0,1)$ is continuous and X is compact. Let $B_0 := u(X), B_1 := \underline{b}\, u(X)$, and $B_2 := \underline{b}^2 u(X)$. Note that $B_0 \times B_1 \times B_2$ is a rectangle in $\mathbb{R}^3$ with nonempty interior. We wish to show that $B_0 \times B_1 \times B_2 \subset C$. Take any $(v_0, v_1, v_2) \in B_0 \times B_1 \times B_2$. Let $y_0 \in X$ be such that $u(y_0) = v_0$. Recall that utility is normalized so that $0 \in u(X)^\circ$. Thus, by construction,

$$v_1 \in B_1 = \underline{b}\, u(X) \subset b(y_0)u(X).$$

Hence, we can find $y_1 \in X$ such that $v_1 = b(y_0)u(y_1)$. Similarly,

$$v_2 \in B_2 = \underline{b}^2 u(X) \subset b(y_0)b(y_1)u(X).$$

Hence, we can find $y_2 \in X$ such that $v_2 = b(y_0)b(y_1)u(y_2)$. Conclude that $B_0 \times B_1 \times B_2 \subset C$. Hence, $L(B_1 \times B_2 \times B_3) \subset \Phi(X^3)$. Since $B_0 \times B_1 \times B_2$ has nonempty interior and L maps open sets into open sets, $\Phi(X^3)^\circ \neq \varnothing$. By construction, $\mathbf{0} \in [B_1 \times B_2 \times B_3]^\circ$ and, hence, $\mathbf{0} \in \Phi(X^3)^\circ$. ∎

Next, consider the mapping

$$\Gamma(y_0, y_1, y_2) := \frac{1}{1 - \hat{b}(y_0, y_1, y_2)} \Phi(y_0, y_1, y_2).$$

Note that $\Gamma(X^3) = U \circ \mathcal{H}_3^{rp}$ and recall that we identify $U \circ \mathcal{H}_3^{rp}$ with a subset of $\mathbb{R}^3$. Focus on the restriction of I to $\mathbb{R}^3$. The remainder of the proof is broken in two cases.

Case 1: Suppose $b : X \to (0,1)$ is constant so that $b(x) = \beta$ for every $x \in X$. Then, $\Gamma = (1 - \beta^{12})^{-1}\Phi$, implying that $\Gamma(X^3)$ has nonempty interior in $\mathbb{R}^3$ and $\mathbf{0} \in O := \Gamma(X^3)^\circ$.

Lemma 28 *I is positively homogeneous.*

Proof. Suppose by way of contradiction that I is not positively homogeneous. Then, we can find $\xi \in O$ and $\alpha \in (0,1)$ such that $I(\xi) = 0$ but $I(\alpha\xi) \neq 0$. Since I is continuous, there is $\alpha' \in (\alpha, 1]$ such that $I(\alpha'\xi) = 0$ and $I(\alpha''\xi) \neq 0$ for all $\alpha'' \in [\alpha, \alpha')$. But since $\alpha'\xi \in O$, there is some repeating permutation act $h \in \mathcal{H}_3^{rp}$ such that $U \circ h := \alpha'\xi$. Adopting the notation used in Lemma 26, write h as a sequence $(a, a, ...)$ where a is some finite list of Π-measurable functions $f : \Omega \to X$. Let $g^1 := (a, x_0, a, x_0, ...)$, $g^2 := (a, a, x_0, a, a, x_0, ...)$, and so on. By construction, the sequence $(g^n)_n$ converges pointwise to h. For every n, a direct calculation shows that $U \circ g^n = \alpha^n U \circ h$ for some $\alpha_n \in (0,1)$ and $\alpha^n \nearrow_n 1$. Hence, for some n large enough, $\alpha' \alpha^n \in [\alpha, \alpha']$ so that $I(U \circ g^n) \neq 0$. But Lemma 26 implies that $I(U \circ g^n) = 0$ for every n, a contradiction. ■

Lemma 29 *I is quasiconcave.*

Proof. Suppose not. Since I is positively homogeneous, we can find $\xi, \xi' \in O$ and $\gamma \in (0,1)$ such that $I(\xi) = I(\xi') = 0$ and $I(\gamma\xi + (1-\gamma)\xi') < 0$. Since I is continuous, we can also choose ξ and ξ' such that the preceding inequality obtains for every $\gamma \in (0,1)$. Since $\xi, \xi' \in O = U \circ \mathcal{H}_3^{rp}$, there are repeating permutation acts h and h' such that $U \circ h = \xi$ and $U \circ h' = \xi'$. IHUT implies that $I(\gamma\xi + (1-\gamma)\xi') \geq 0$ for some $\gamma \in (0,1)$, a contradiction. ■

Case 2: Suppose $b : X \to (0,1)$ is nonconstant. Theorem 1 shows that I is positively homogeneous. It remains to show that I is quasiconcave.[21] By way of contradiction, suppose there are $\xi, \xi' \in U \circ \mathcal{H}$ and $\gamma \in (0,1)$ such that $I(\xi) = 0, I(\xi') > 0$, and $I(\gamma\xi' + (1-\gamma)\xi) < 0$. Since I is continuous, we can assume that the inequality holds for all γ in some interval $(0, \underline{\gamma}) \subset (0,1)$. Since I is continuous and $U \circ \mathcal{H}^{rp}$ is dense in $U \circ \mathcal{H}$, we can also assume that $\xi' \in U \circ \mathcal{H}^{rp}$. From Lemma 27, we know that the ray S extending from $\mathbf{0} \in \mathbb{R}^3$ and passing through ξ' intersects the open set $\Phi(X^3)^\circ \ni \mathbf{0}$. Thus, we can find a sequence $(y_0^n, y_1^n, y_2^n)_n$ such that $\Phi(y_0^n, y_1^n, y_2^n) \in S$ for every n and $\Phi(y_0^n, y_1^n, y_2^n) \to_n \mathbf{0} \in \mathbb{R}^3$. Let

$$\xi_n := \frac{1}{1 - \hat{b}(y_0^n, y_1^n, y_2^n)} \Phi(y_0^n, y_1^n, y_2^n)$$
$$\lambda_n := 1 - \hat{b}(y_0^n, y_1^n, y_2^n)$$

Since $b(X) \subset (0,1)$, the λ_n are bounded away from 0 and 1, that is, there is $\varepsilon > 0$ such that $\lambda_n \in [\varepsilon, 1-\varepsilon] \subset (0,1)$ for each n. Since $\Phi(y_0^n, y_1^n, y_2^n) \to_n \mathbf{0}$, it follows that $\xi_n \to_n \mathbf{0}$. By construction, each ξ_n lies on the ray through ξ'. Hence, $\xi_n = k_n \xi'$ for some $k_n > 0$. Also, $k_n \to_n 0$. Since I is positively homogeneous, we know that $I(\xi_n) = k_n I(\xi') > 0 = I(\xi)$. For each n, there is a repeating permutation act $h_n \in \mathcal{H}_3^{rp}$

[21] If one could once again show that $\Gamma(X^3)$ has nonempty interior, then Lemma 29 would deliver the desired result. When $b : X \to (0,1)$ is non-constant however, we have not been able to prove that the interior of $\Gamma(X^3)$ is nonempty. Hence, we provide a different argument for the quasiconcavity of I.

such that $U \circ h_n = \xi_n$. By IHUT,

$$0 \leq I(\lambda_n \xi_n + (1-\lambda_n)\xi) = I(\lambda_n k_n \xi' + (1-\lambda_n)\xi) \quad \forall n. \tag{A.21}$$

Let $\alpha_n := \frac{\lambda_n k_n}{\lambda_n k_n + (1-\lambda_n)}$. Since I is positively homogeneous, (A.21) implies that

$$0 \leq I(\alpha_n \xi' + (1-\alpha_n)\xi) \quad \forall n. \tag{A.22}$$

Recall that $k_n \to_n 0$, while the sequence $(\lambda_n)_n$ is bounded away from 1. Thus, $\alpha_n \to_n 0$. But then (A.22) contradicts the fact that $I(\gamma\xi' + (1-\gamma)\xi) < 0$ for all γ in the open interval $(0, \underline{\gamma})$.

References

[1] C. Aliprantis and K. Border. *Infinite Dimensional Analysis*. Springer-Verlag, 2 edition, 1999. A.1.3

[2] S. Alon and D. Schmeidler. Purely subjective maxmin expected utility. *Journal of Economic Theory*, 152:382–412, 2014. 7.1

[3] D. K. Backus, B. R. Routledge, and S. E. Zin. Exotic preferences for macroeconomists. In *NBER Macroeconomics Annual*, volume 19, pages 319–390. MIT Press, 2004. 8

[4] A. Bommier, A. Kochov, and F. LeGrand. On monotone recursive preferences. forthcoming in Econometrica, 2016. 3.1, 7.2, 7.2, 7.3, 20

[5] R. Casadesus-Masanell, P. Klibanoff, and E. Ozdenoren. Maxmin expected utility over Savage acts with a set of priors. *Journal of Economic Theory*, 92 (1):35–65, 2000. 7.1

[6] S. H. Chew and L. G. Epstein. Recursive utility under uncertainty. In A. M. Khan and N. C. Yannelis, editors, *Equilibrium Theory in Infinite Dimensional Spaces*, volume 1 of *Studies in Economic Theory*, pages 352–369. Springer Berlin Heidelberg, 1991. 7.3

[7] G. Debreu. Representation of a preference ordering by a numerical utility function. In R. Thrall, C. Coombs, and R. Davis, editors, *Decision Processes*. John Wiley & Sons, 1954. A.1.2

[8] D. Ellsberg. Risk, ambiguity and the Savage axioms. *Quarterly Journal of Economics*, 75:643–669, 1961. 1

[9] L. G. Epstein. Stationary cardinal utility and optimal growth under uncertainty. *Journal of Economic Theory*, 31(1):133–152, 1983. 1, 3.1, 3.2, 4, 5, 6, 7.3

[10] L. G. Epstein. A simple dynamic general equilibrium model. *Journal of Economic theory*, 41(1):68–95, 1987. 8

[11] L. G. Epstein and J. A. Hynes. The rate of time preference and dynamic economic analysis. *The Journal of Political Economy*, 91(4):611–635, 1983. 1

[12] L. G. Epstein and M. Schneider. Ambiguity and asset markets. *Annual Review of Financial Economics*, 2:315–346, 2010. 1

[13] L. G. Epstein and S. Zin. Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework. *Econometrica*, 57 (4):937–969, 1989. 7.3

[14] P. Ghirardato and M. Marinacci. Risk, ambiguity, and the separation of utility and beliefs. *Mathematics of operations research*, 26(4):864–890, 2001. 7.1

[15] P. Ghirardato and M. Marinacci. Ambiguity made precise: A comparative foundation. *Journal of Economic Theory*, 102(2):251–289, 2002. 3.1

[16] P. Ghirardato, F. Maccheroni, M. Marinacci, and M. Siniscalchi. A subjective spin on roulette wheels. *Econometrica*, 71:1897–1908, 2003. 7.1

[17] P. Ghirardato, F. Maccheroni, and M. Marinacci. Differentiating ambiguity and ambiguity attitude. *Journal of Economic Theory*, 118(2):133–173, 2004. 1

[18] I. Gilboa and D. Schmeidler. Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics*, 18:141–153, 1989. 1, 7.1

[19] A. Kochov. Time and No Lotteries: An Axiomatization of Maxmin Expected Utility. *Econometrica*, 83(1):239–262, 2015. 1, 3, 3.1, 3.2, 4, 7.1, 7.3, A.1.1, A.1.9

[20] A. Kochov. A separation result for stationary preferences. *Journal of Mathematical Economics*, 70:123–126, 2017. 11

[21] T. Koopmans. Stationary ordinal utility and impatience. *Econometrica*, 28(2): 287–309, 1960. 1, 3.1, 4, 7.2

[22] T. C. Koopmans, P. A. Diamond, and R. E. Williamson. Stationary utility and time perspective. *Econometrica*, pages 82–100, 1964. 4

[23] D. Kreps and E. Porteus. Temporal resolution of uncertainty and dynamic choice theory. *Econometrica*, 46(1):185–200, 1978. 7.4

[24] R. Lucas and N. Stokey. Optimal growth with many consumers. *Journal of Economic Theory*, 32(1):139–171, 1984. 8

[25] A. Lundberg. Cliff functions-a tool tor treating sequences of composed functions. *Aequationes Mathematicae*, 18:35–53, 1978. A.1.4

[26] A. Lundberg. Generalized distributivity for real, continuous functions. i: Structure theorems and surjective solutions. *Aequationes Mathematicae*, 24(1):74–96, 1982. 7.2, 7.2, 7.3, A.1.3, A.1.3, A.1.4, A.1.4, A.1.4, A.1.4, A.1.5, A.1.5

[27] A. Lundberg. Generalized distributivity for real, continuous functions. ii: Local solutions in the continuous case. *Aequationes Mathematicae*, 28(1):236–251, 1985. A.1.8, A.1.8, A.1.9, A.1.10

[28] F. Maccheroni, M. Marinacci, and A. Rustichini. Dynamic variational preferences. *Journal of Economic Theory*, 128(1):4–44, 2006. 7.1, 7.4

[29] H. Royden. *Real Analysis*. Macmillan, New York, 3rd edition, 1988. A.2

[30] L. Savage. *The Foundations of Statistics*. Wiley, New York, NY, 1954. 2, 4

[31] T. Strzalecki. Temporal resolution of uncertainty and recursive models of ambiguity aversion. *Econometrica*, 81(3):1039–1074, 2013. 7.4

[32] H. Uzawa. Time preference, the consumption function, and optimum asset holdings. In *Value, Capital and Growth*. Aldine, Chicago, 1968. 1