Perron-Frobenius theory recovers more than what you think:
The example of limited participation

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Abstract

In a seminal article, Hansen and Scheinkman (2009) have proved that Perron-Frobenius theory helps recover a probability measure that can be used for pricing long-term claims. In this paper, we show that the recovered probability also contains information about the market structure. More precisely, we provide an example, in which applying Perron-Frobenius theory enables to measure the degree of limited market participation.

Keywords: Perron-Frobenius, Arrow-Debreu securities, limited participation.

1 Introduction

Asset prices contain information about both stochastic discount factors and transition probabilities. Extracting this information using Perron-Frobenius (PF, henceforth) theory has been pioneered by Backus, Gregory, and Zin (1989). More precisely, Hansen

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and Scheinkman (2009) have proved that applying PF theory to Arrow-Debreu (AD, henceforth) security prices enables to recover a probability measure, which provides useful insights for the pricing of long-term claims. As shown in Ross (2015), the recovered probability is equal to the actual one under some specific conditions. Borovička, Hansen, and Scheinkman (2016) have generalized Ross result and have proven that in general, the recovered probability differs from the actual one by a martingale component, which is trivial only under Ross conditions. Interestingly, most asset pricing models in the economic literature feature a non-trivial martingale component.

In this paper, we show in an example that applying PF theory enables to recover information not only about long-term pricing but also about market structure and limited market participation in particular. Our example features an economy populated by two heterogeneous agents trading AD securities. Agent heterogeneity generates an endogenous market segmentation. The same agent does not trade all securities in all states of the world and the market arrangement is not the same for all maturities. Applying PF theory delivers two main results in this context. First, the long-term return recovered by the PF theory differs from the actual one due to limited market segmentation. The largest eigenvalue of the matrix of AD prices now reflects a long-term discount rate that is distorted by limited participation and agents' heterogeneity. Second, the recovered and the actual long-term one-period expected holding returns also differ from each other. The differences between actual and recovered returns –for both long-term returns and one-period expected holding returns– are proved to be monotonic functions of agent heterogeneity. Consequently, PF theory can help measure agent heterogeneity and quantify the severity of limited market participation. In addition to be useful for long-term pricing, PF also proves to be insightful for quantifying the degree of market segmentation. Up to our knowledge, this is the first paper showing the link between PF theory and financial market structure.

The rest of the paper is organized as follows. The set-up of the economy is presented in Section 2. The implications of PF theory are derived in Section 3. Section 4 concludes.

2 Set-up

We consider an economy populated by two agents denoted A and B. There are two states of the world denoted 1 and 2. The process determining states of the world follows a first-order Markov chain characterized by probabilities π and ν, that are the probabilities to remain in states 1 and 2, respectively. Agents can trade two AD securities. The price in state $i = 1, 2$ of the AD security paying off in state $j = 1, 2$ of the next period is denoted q_{ij}. The pricing kernel v_{ij}^k of agent $k = A, B$ in state $i = 1, 2$ for a payoff in state $j = 1, 2$ of the next period is assumed to be equal to:

$$v_{ij}^k = \beta^k \frac{m_i^k}{m_j^k}. \tag{1}$$

We introduce two sources of heterogeneity among agents: (i) the coefficients β^k and (ii) the ratios $\frac{m_1^k}{m_2^k}$ (or equivalently $\frac{m_2^k}{m_1^k}$) for $k = A, B$. These two sources are the minimal ingredients to generate a nontrivial market structure, that is furthermore not the same for securities with different maturities. Such a set-up could for instance result from a no–trade equilibrium in an economy featuring heterogeneous agents (in β and in endowments), credit constraints and zero net asset supply. See for instance **?** or Challe, LeGrand, and Ragot (2013) for similar set-ups with bond pricing.

The framework of this example is a slight deviation from the recovery setup (see Borovička, Hansen, and Scheinkman, 2016; Ross, 2015). Note that Ross recovery result holds in our economy when heterogeneity is absent, i.e., when $\beta^A = \beta^B$ and $\frac{m_1^A}{m_2^A} = \frac{m_1^B}{m_2^B}$.

We introduce the following notation:

$$\tau^m = \frac{\beta^A}{\beta^B} \frac{m_2^B}{m_1^B} \frac{m_1^A}{m_2^A}, \quad \tau^\beta = \frac{\beta^A}{\beta^B}. \tag{2}$$

The two quantities τ^m and τ^β summarize the two dimensions of heterogeneity and measure the deviation of the actual economy to the recovery setup, which corresponds to $\tau^\beta = \tau^m = 1$. We furthermore make the following assumption:

Assumption A *We assume that* $\tau^\beta \leq 1$ *and* $\tau^m \geq 1$.

The first part of Assumption A ($\tau^\beta \leq 1$) in fact means that agent B is more patient than agent A. The second part ($\tau^m \geq 1$) states that the marginal rate of substitution between state 2 now and state 1 in the next period is larger for agent A than for agent B.

Assumption A enables to generate endogenous limited participation and a nontrivial market structure. Indeed, it implies that the AD security paying off in state 2 is always traded by agent B, while the other AD security paying off in state 1 is traded by agent A in state 2, and by B in state 1. We deduce that the matrix $Q = (q_{ij})_{i,j=1,2}$ of AD security prices can be expressed as:

$$Q = \beta^B \begin{pmatrix} \pi & (1-\pi)\frac{m_2^B}{m_1^B} \\ (1-\nu)\tau^m \frac{m_1^B}{m_2^B} & \nu \end{pmatrix}. \qquad (3)$$

3 Recovering limited participation from Perron-Frobenius theory

We analyze the application of the PF theory. In contrast with existing literature, we prove that the long-term return recovered from one-period AD security prices differs from the actual long-term return and that the difference between both is monotone with agents' heterogeneity. Interestingly, PF theory enables to recover the severity of market segmentation and in particular its determinants, which are the coefficients τ_β and τ_m.

In the remainder of the paper, we consider AD securities paying off in n periods. The price in state i of an AD security paying off in state j in n periods is denoted $q_{ij}^{(n)}$.

3.1 A preliminary lemma

For avoiding discussing too many cases, we make the following assumption:

Assumption B *We assume that agents pricing kernels are such that:*

$$m_B^1 \geq m_B^2,$$
$$(\tau^m - 1)\,\pi \geq \nu(1-\tau^\beta),$$
$$(1-\nu)(1-\pi)\,(\tau^m - 1) \geq \nu^2(1-\tau^\beta)\tau^\beta.$$

Assumption B is compatible with Assumption A and holds when the ratio $\frac{m_1^A}{m_2^A}$ is sufficiently large, or when the ratio τ^β is sufficiently close to one. This condition can be interpreted as the fact that the heterogeneity due to $\frac{m_2^k}{m_1^k}$ is "stronger" than the one due to β^k.

Assumption B looks involved but is simply meant to generate a non-trivial market structure for AD securities with a maturity greater than 2 periods. The next lemma summarizes the market structure implied by Assumption B.

Lemma 1 (Market structure) *If Assumption B holds, for any AD security of maturity* $n \geq 2$*, the agent* A *is price-maker in state 2, while agent* B *is price-maker in state 1.*

Proof of Lemma 1 can be found in Section A of the Appendix. Assumption B implies a perfect market segmentation for AD securities with maturity greater than two periods. Only agent A trades in state 2, while only agent B trades in state 1. The market segmentation for securities with maturity greater than two periods differs from the market arrangement observed for one-period AD securities. This difference in market segmentation for AD securities with different maturities is key to explain why the limited participation affects the long-term rate recovered from the PF theory.

3.2 First dimension: Recovering τ^β

We now state our first result about the impact of limited market participation on long-term asset returns.

Proposition 1 (Limited market participation) *The actual long-term return* r_∞^a *differs from the one recovered from one-period AD prices by the PF theory,* r_∞^{PF}.

For any τ^m, the difference between both long-term returns, denoted $\delta r_\infty = r_\infty^{PF} - r_\infty^a$, is always nonpositive and increases with τ^β. Furthermore:

- *when $\tau^\beta \to 1$, $\delta r_\infty \to 0$;*
- *when $\tau^\beta \to 0$, $\delta r_\infty \to \ln\left(\frac{\pi+\left(\pi^2+4(1-\pi)(1-\nu)\tau^m\right)^{\frac{1}{2}}}{\pi+\nu+((\pi-\nu)^2+4(1-\pi)(1-\nu)\tau^m)^{\frac{1}{2}}}\right) < 0$.*

The proof can be found in Section B of the Appendix.

Proposition 1 shows that applying the PF theory to one-period AD security prices does not yield actual long-term return. More precisely, the proposition guarantees the existence of a one-to-one relationship between the dimension τ^β of limited market participation and the long-term return gap δr_∞. Therefore, the difference between actual and recovered returns indirectly enables to recover the dimension τ^β of the limited market participation, while keeping the dimension τ^m unchanged.

The intuition for the result of Proposition 1 can be understood from the PF theory. Indeed, the long-term rate is usually characterized by the largest eigenvalue of the matrix of one-period AD prices (here, Q). However, due to the market structure described in Lemma 1, this does not hold in our economy. The long-term rate is determined by the largest eigenvalue of another matrix, that differs from Q.

We now show how to recover the second dimension of limited market participation.

3.3 Second dimension: Recovering τ^m

We now take advantage of the long-term one-period expected holding returns to recover the heterogeneity in τ^m. We define the one-period expected holding return (EHR henceforth) for a zero-coupon of maturity n as follows. The price in state i of the n-period zero-coupon bond is $p_i^{(n)} = q_{i1}^{(n)} + q_{i2}^{(n)}$. The EHR $r_{i,(n)}^{1,\Pi}$ is equal to the expected return for purchasing a n-period bond in state i and reselling it in one period as a $(n-1)$-period bond. More formally, we have:

$$r_{i,(n)}^{1,\Pi} = \frac{\Pi_{i1} p_1^{(n-1)} + \Pi_{i2} p_2^{(n-1)}}{p_i^{(n)}}.$$

The average EHR $r^{1,\Pi}_{(n)}$ is equal to the unconditional average of state-dependent EHR:

$$r^{1,\Pi}_{(n)} = \frac{1-\Pi_{22}}{2-\Pi_{11}-\Pi_{22}} r^{1,\Pi}_{1,(n)} + \frac{1-\Pi_{11}}{2-\Pi_{11}-\Pi_{22}} r^{1,\Pi}_{2,(n)}.$$

In the rest of the paper, we denote $r^1_{(n)}$ the EHR when the transition probabilities are actual probabilities, while $\widetilde{r}^1_{(n)}$ corresponds to the EHR when the transition probabilities are recovered by PF theory. We also note r^1_∞ and $\widetilde{r}^1_\infty$ their respective limits when the maturity n becomes very large and converges to infinity. We now state the following proposition.

Proposition 2 (One-period holding return) *The ratio of one period expected holding returns,* $\frac{\widetilde{r}^1_\infty}{r^1_\infty}$ *is an increasing function of* τ^m*. In particular,* $\widetilde{r}^1_\infty \geq r^1_\infty$.

The proof can be found in Section C of the Appendix. Proposition 2 shows that the heterogeneity in τ^m raises the difference between EHR under the recovered and the actual probability measures. Moreover, the recovered EHR is always greater than the actual one: the recovered probability puts greater weight on larger returns than it is actually the case. Finally, Proposition 2 parallels Proposition 1 and shows that EHRs enable to recover the dimension τ^m of agents' heterogeneity, while keeping the other dimension τ^β unchanged.

3.4 Recovering the determinants of market segmentation

Propositions 1 and 2 have established partial invertibility results stating that the observation of either long-term returns or long-term one-period holding returns was sufficient for recovering one of the heterogeneity determinants τ^β or τ^m. We now state a global invertibility result showing that observing both long-term returns and long-term EHR enable to jointly recover both determinants of market segmentation.

Proposition 3 (Recovering (τ^β, τ^m)) *For a given observation of long-term return difference* $r^{PF}_\infty - r^a_\infty$ *and of long-term EHR ratio* $\frac{\widetilde{r}^1_\infty}{r^1_\infty}$*, there corresponds a unique pair* (τ^β, τ^m).

The proof can be found in Section B of the Appendix. This result shows that the application of the PF theory enables to completely quantify limited financial participation in our example economy. It generalizes the results of Propositions 1 and 2 to the joint recovery of the determinants of market segmentation.

4 Conclusion

We have shown that the application of the PF theory in a context of limited market participation only allows one to obtain a distorted version of the long term return. However, PF theory enables to recover the determinants of limited market participation and understand the underlying heterogeneity of market participants. Up to our knowledge, this paper is the first one to illustrate the possible connection between PF and financial market structure. We leave the exploration of the general theory of PF with limited participation for future research.

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Appendix

A Proof of Lemma 1

We introduce the two following notations:

$$Q_A = \beta^B \begin{pmatrix} \pi & (1-\pi)\frac{m_2^B}{m_1^B} \\ (1-\nu)\frac{m_1^B}{m_2^B}\tau^m & \nu\tau^\beta \end{pmatrix} \text{ and } Q_B = \beta^B \begin{pmatrix} \pi & (1-\pi)\frac{m_2^B}{m_1^B} \\ (1-\nu)\frac{m_1^B}{m_2^B} & \nu \end{pmatrix}. \quad (4)$$

The matrix Q_A collects (possibly out-of-equilibrium) AD prices when B is price-maker in state 1 and A price maker in state 2. Similarly, the matrix Q_B is the matrix of AD prices when B is price-maker in both states. For Lemma 1 to hold, $(Q_A - Q_B)Q_A^n Q$ must have only positive elements for all n. Diagonalizing Q_A yields $Q_A = E_A D_A E_A^{-1}$ with:

$$\begin{aligned}
D_A &= \mathrm{diag}(K_1(\tau^\beta, \tau^m), K_2(\tau^\beta, \tau^m)), \\
E_A &= \beta^B \begin{pmatrix} (1-\pi)\frac{m_1^B}{m_2^B} & (1-\pi)\frac{m_1^B}{m_2^B} \\ \frac{1}{2}\left(\nu\tau^\beta - \pi + \sqrt{\Delta(\tau^\beta, \tau^m)}\right) & \frac{1}{2}\left(\nu\tau^\beta - \pi - \sqrt{\Delta(\tau^\beta, \tau^m)}\right) \end{pmatrix}, \\
\Delta(\tau^\beta, \tau^m) &= (\pi - \nu\tau^\beta)^2 + 4(1-\pi)(1-\nu)\tau^m, \\
K_i(\tau^\beta, \tau^m) &= \frac{\beta^B}{2}\left(\pi + \nu\tau^\beta + (-1)^{i-1}\sqrt{\Delta(\tau^\beta, \tau^m)}\right), \ i = 1, 2,
\end{aligned}$$

Dropping the dependence in (τ^β, τ^m), we have after some algebra $(Q_A - Q_B)Q_A^n Q = \begin{pmatrix} 0 & 0 \\ d_{1,n} & d_{2,n} \end{pmatrix}$ for all $n \geq 0$, where:

$$\begin{aligned}
\frac{\frac{m_1^A}{m_2^A} d_{1,n}}{\beta^A\beta^B(1-\nu)} &= \left((1 - 1/\tau^m)\,\pi - \nu(1-\tau^\beta)\right)(\lambda_{A,1}^{n+1} - \lambda_{A,2}^{n+1}) && (5)\\
&\quad + \beta^A\left((1-\pi)(1-\nu)\tau^\beta\,(\tau^m - 1) - \nu\pi\,(1 - 1/\tau^m)\right)(\lambda_{A,1}^n - \lambda_{A,2}^n), \\
\frac{d_{2,n}}{(\beta^B)^2} &= \left((1-\nu)(1-\pi)\,(\tau^m - 1) - \nu\nu(1-\tau^\beta)\right)(\lambda_{A,1}^{n+1} - \lambda_{A,2}^{n+1}) && (6)\\
&\quad + \nu(1-\tau^\beta)\,(\nu + \pi - 1)\,\beta^B(\lambda_{A,1}^n - \lambda_{A,2}^n).
\end{aligned}$$

Before going further, let us remark that:

$$\lambda_{A,1}^{n+1} - \lambda_{A,2}^{n+1} \geq \nu\beta^A(\lambda_{A,1}^n - \lambda_{A,2}^n), \tag{7}$$

$$\geq \pi\beta^B(\lambda_{A,1}^n - \lambda_{A,2}^n). \tag{8}$$

Let us prove (7) (the proof for (8) is similar). Note that (i) $\lambda_{A,2} < \lambda_{A,1}$ and (ii) $0 \leq \nu\beta^A \leq \lambda_{A,1}$. First, since $\lambda_{A,1} > 0$, (7) is equivalent to $1 - \frac{\nu\beta^A}{\lambda_{A,1}} \geq \frac{\lambda_{A,2}^n}{\lambda_{A,1}^n}(\frac{\lambda_{A,2}}{\lambda_{A,1}} - \frac{\nu\beta^A}{\lambda_{A,1}})$. The result holds when$\lambda_{A,2} \geq 0$ or when $\lambda_{A,2} < 0$ and n is even. We now assume that $n = 2m+1$ and $\lambda_{A,2} < 0$. The sequence $m \mapsto \frac{\lambda_{A,2}^{2m+1}}{\lambda_{A,1}^{2m+1}}(\frac{\lambda_{A,2}}{\lambda_{A,1}} - \frac{\nu\beta^A}{\lambda_{A,1}})$ is positive and decreasing since $\frac{\lambda_{A,2}^2}{\lambda_{A,1}^2} \in [0,1)$. We conclude by proving that (7) holds for $m = 0$. Indeed, it is equivalent to $\lambda_{A,1} + \lambda_{A,2} \geq \nu\beta^A$, which holds since $\lambda_{A,1} + \lambda_{A,2} = \pi\beta^B + \nu\beta^A$.

Now, using (5) and (7), $d_{1,n}$ becomes $\frac{d_{1,n}}{\beta^A\beta^B(1-\nu)(\lambda_{A,1}^n - \lambda_{A,2}^n)\beta^A\frac{m_2^A}{m_1^A}} \geq (1-\pi)(1-\nu)\left(\tau^m - 1\right)/\tau^\beta - \nu^2(1-\tau^\beta) \geq 0$, where the second inequality comes from Assumption B. We similarly prove that $d_{2,n} \geq 0$, which concludes the proof.

B Proof of Proposition 1

The largest eigenvalue of one-period AD securities is $\lambda_Q = \frac{\beta^B}{2}\left(\pi + \nu + \Delta^{1/2}(1,\tau^m)\right)$. However, due to the market structure of Lemma 1, the long-term return of an AD security depends on $\lambda_{Q_A} = \frac{\beta^B}{2}\left(\pi + \nu\tau^\beta + \Delta^{1/2}\right)$. The actual long-term rate is thus $r_\infty^a = -\log(\lambda_{Q_A})$, while the recovered one is $r_\infty^{PF} = -\log(\lambda_Q)$. The difference of both rates is $\delta r_\infty = r_\infty^{PF} - r_\infty^a = \log\left(\frac{\lambda_{Q_A}}{\lambda_Q}\right)$, where:

$$\frac{\lambda_{Q_A}}{\lambda_Q} = \frac{\pi + \nu\tau^\beta + \left((\pi - \nu\tau^\beta)^2 + 4(1-\pi)(1-\nu)\tau^m\right)^{\frac{1}{2}}}{\pi + \nu + \left((\pi-\nu)^2 + 4(1-\pi)(1-\nu)\tau^m\right)^{\frac{1}{2}}}. \tag{9}$$

We denote δr_∞ as $\varphi(\tau^\beta, \tau^m)$, with:

$$\frac{\partial\varphi(\tau^\beta,\tau^m)}{\partial\tau^\beta} = -C \times \left(1 - (\pi - \nu/\tau^\beta)\left((\pi - \nu/\tau^\beta)^2 + 4(1-\pi)(1-\nu)\tau^m\right)^{-\frac{1}{2}}\right), \tag{10}$$

where $C > 0$. We deduce that δr_∞ increases with τ^β and since $\varphi(1, \tau^m) = 1$, $\delta r_\infty \leq 0$.

C Proof of Proposition 2

PF theory for AD prices in matrix Q of (3) implies that the recovered probability $\widetilde{\mathbb{P}}$ is characterized by the transition matrix $\widetilde{P}$:

$$\widetilde{P} = \begin{pmatrix} \widetilde{\pi} & 1-\widetilde{\pi} \\ 1-\widetilde{\nu} & \widetilde{\nu} \end{pmatrix}, \tag{11}$$

$$\text{where: } \widetilde{\pi} = \frac{2\pi}{\pi + \nu + \sqrt{\Delta_1}}, \ \widetilde{\nu} = \frac{2\nu}{\pi + \nu + \sqrt{\Delta_1}}, \text{ and } \Delta_1 = \Delta(1, \tau_m). \tag{12}$$

The price $p_i^{(n)}$ in state $i = 1, 2$ of a zero-coupon bond of maturity n periods is:

$$p_i^{(n)} = \lambda_{Q_A}^{n-1} \frac{\beta^B}{\sqrt{\Delta}} \rho_i (1 + o_n(1)), \quad i = 1, 2, \tag{13}$$

$$\rho_1 = \frac{\pi - \nu\tau^\beta + \sqrt{\Delta}}{2}\pi + (1-\pi)(1-\nu)\tau^m + (1-\pi)\frac{m_2^B}{m_1^B}\frac{\pi + \nu(2-\tau^\beta) + \sqrt{\Delta}}{2}, \tag{14}$$

$$\rho_2 = \frac{-\pi + \nu\tau^\beta + \sqrt{\Delta}}{2}\nu + (1-\pi)(1-\nu)\tau^m + (1-\nu)\tau^m \frac{m_1^B}{2m_2^B}\left(\pi + \nu\tau^\beta + \sqrt{\Delta}\right), \tag{15}$$

where $o_n(1) \rightarrow_{n\to\infty} 0$. After some algebra, (14) becomes:

$$\rho_1 = \frac{1}{2}\left(-\pi + \nu\tau^\beta + \sqrt{\Delta_A}\right)\frac{m_1^B}{(1-\pi)m_2^B}\rho_2. \tag{16}$$

We denote $r_{(n)}^1$ the one-period average EHR for a n-period bond under (actual) $\mathbb{P}$, while $\widetilde{r}_{(n)}^1$ denotes the same return under (recovered) $\widetilde{\mathbb{P}}$. Using (13), we obtain:

$$r_{(n)}^1 = \frac{2}{\beta((\pi+\nu) + \sqrt{\Delta})}\left(1 + \frac{(1-\pi)(1-\nu)}{2-\pi-\nu}\frac{(\rho_1-\rho_2)^2}{\rho_1\rho_2}\right) + o_n(1), \tag{17}$$

$$\tilde{r}_{(n)}^1 = \frac{2}{\beta((\pi+\nu) + \sqrt{\Delta})}\left(1 + \frac{(1-\widetilde{\pi})(1-\widetilde{\nu})}{2-\widetilde{\pi}-\widetilde{\nu}}\frac{(\rho_1-\rho_2)^2}{\rho_1\rho_2}\right) + o_n(1). \tag{18}$$

We deduce from (17) and (18) $\frac{\tilde{r}^1_{(n)}}{r^1_{(n)}} = 1 + \frac{(1-\widetilde{\pi})(1-\widetilde{\nu})}{2-\widetilde{\pi}-\widetilde{\nu}} \frac{(\rho_1-\rho_2)^2}{\rho_1\rho_2} \Big/ 1 + \frac{(1-\pi)(1-\nu)}{2-\pi-\nu} \frac{(\rho_1-\rho_2)^2}{\rho_1\rho_2}$, and:

$$\frac{\tilde{r}^1_{(n)}}{r^1_{(n)}} = \psi(\tau^m) = \frac{1+g(\tau^m)f(\tau^m)}{1+f(\tau^m)}, \tag{19}$$

$$\text{where: } g(\tau^m) = \frac{2(2-\pi-\nu)\tau^m}{\sqrt{\Delta(\tau^m)}(\pi+\nu+\sqrt{\Delta(\tau^m)})}, \tag{20}$$

$$f(\tau^m) = \frac{(1-\pi)(1-\nu)}{2-\pi-\nu}\left(\frac{\rho_1(\tau^m)}{\rho_2(\tau^m)} + \frac{\rho_2(\tau^m)}{\rho_1(\tau^m)} - 2\right). \tag{21}$$

Using (19) we have:

$$\psi'(\tau^m) = \frac{g'(\tau^m)f(\tau^m)(1+f(\tau^m)) + f'(\tau^m)(g(\tau^m)-1)}{(1+f(\tau^m))^2}. \tag{22}$$

If g and f are increasing, we have $g(\tau^m) \geq 1$ for any $\tau^m \geq 1$ since $g(1) = 1$. Since $g, f > 0$, (22) implies $\psi'(\tau^m) > 0$ for any $\tau^m \geq 1$ and ψ strictly increasing on $[1, \infty)$. This also implies $\psi(\tau^m) \geq 1$ and $\tilde{r}^1_{(n)} \geq r^1_{(n)}$.

We now show that g is increasing. From (20):

$$-\frac{\partial}{\partial\tau^m}\ln(g(\tau^m)) = \frac{\Delta'(\tau^m)}{2\Delta(\tau^m)} + \frac{\Delta'(\tau^m)}{2\sqrt{\Delta(\tau^m)}(\pi+\nu+\sqrt{\Delta(\tau^m)})} - \frac{1}{\tau^m}.$$

Eq. (12) yields $-\frac{\partial}{\partial\tau^m}\ln(g(\tau^m)) \leq \frac{-(\pi-\nu)^2}{\tau^m\Delta(\tau^m)} < 0$, which implies that g is increasing.

We now consider the case of f. Using (16), we have:

$$f(\tau^m) = \frac{(1-\pi)(1-\nu)}{2-\pi-\nu}\left(f_1(\tau^m) + \frac{1}{f_1(\tau^m)} - 2\right), \tag{23}$$

$$\text{where: } f_1(\tau^m) = \frac{1}{2}\left(-\pi + \nu/\tau^\beta + \sqrt{\Delta_A}\right)\frac{m_1^B}{(1-\pi)m_2^B}. \tag{24}$$

Since $f > 0$, we deduce from (23) that the sign of f' is given by the sign of $f_1'(\tau^m)(f_1(\tau^m)^2 - 1)$. From (12) and (24), we deduce that $f_1'(\tau^m) > 0$. This implies that $f_1(\tau^m) \geq 1$ for any $\tau^m \geq 1$, since $f_1(1) \geq 1$ because of Assumption B. We deduce that f is increasing, which concludes the proof.

D Proof of Proposition 3

We assume that we observe the long-term return difference δr_∞ and of long-term EHR ratio $\frac{\tilde{r}^1_{(n)}}{r^1_{(n)}}$. From Propositions 1 and 2, we know that there exist $a, b > 0$ such that $\delta r_\infty = e^{-b}$ and $\frac{\tilde{r}^1_{(n)}}{r^1_{(n)}} = e^b$. Using (9) and (19), a pair (τ^β, τ^m) matches observed returns if:

$$\begin{cases} \varphi(\tau^\beta, \tau^m) & = e^{-b}, \\ \psi(\tau^\beta, \tau^m) & = e^a. \end{cases} \tag{25}$$

We now prove that at most one pair (τ^β, τ^m) solves (25). Substituting expressions of φ and ψ, (25) is equivalent to:

$$\begin{cases} \pi + \nu\tau^\beta + \Delta(\tau^\beta, \tau^m) & = e^{-b}\left(\pi + \nu + \Delta(1, \tau^m)\right), \\ \frac{1+g(\tau^m)f(\tau^\beta,\tau^m)}{1+f(\tau^\beta,\tau^m)} & = e^a. \end{cases} \tag{26}$$

where f and f_1 in (23) and (24) are generalized to two variables. Let us define $\hat{f}_1(\tau^m) = \frac{1}{2}\left(e^{-b}\left(\pi + \nu + \Delta(1, \tau^m)\right) - 2\pi\right)\frac{m_1^B}{(1-\pi)m_2^B}$ and $\hat{f}(\tau^m) = \frac{(1-\pi)(1-\nu)}{2-\pi-\nu}\left(\hat{f}_1(\tau^m) + \frac{1}{\hat{f}_1(\tau^m)} - 2\right)$. By construction, $\hat{f}_1$ and $\hat{f}$ are increasing functions. We deduce that (25) implies:

$$\frac{1 + g(\tau^m)\hat{f}(\tau^m)}{1 + \hat{f}(\tau^m)} = e^a, \tag{27}$$

where the left hand-side is increasing (same as for ψ defined in (19)). The solution τ^m of (27) is thus unique (if exists). We also know from Proposition 1 that the solution in τ^β of $\varphi(\tau^\beta, \tau^m) = e^{-b}$ for any τ^m is unique. Thus, the solution (τ^β, τ^m) to (25) is unique.