Risk Aversion and Precautionary Savings in Dynamic Settings

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Abstract

We study the saving behavior of infinitely long-lived agents who face income uncertainty and deterministic interest rates. Using monotone recursive preferences, we prove that risk aversion unambiguously increases savings. The result accounts for possibly binding borrowing constraints and holds for very general specification of income uncertainty, which can follow any kind of stochastically monotone process.

Keywords: risk aversion, precautionary savings, recursive models, risk-sensitive preferences, monotonicity.
JEL codes: D80, D91, E21.

1 Introduction

Analyzing the impact of risk on individual behaviors is a long-standing line of research in decision sciences. However, understanding how individuals make decisions in real life situations is a very difficult question to tackle. Indeed, choices under uncertainty may depend on a number of factors, such as risk appetite, the type of decision that has to be made, interaction between the various risks at stake or insurance possibilities. Clear-cut characterizations of individual behavior under risk can only be ascertained when the problem is more precisely specified.

One particular issue is the so-called precautionary saving problem, where one seeks to understand how future income uncertainty affects saving decisions. The early works of Leland (1968) and Sandmo (1970) on the issue have been followed by numerous contributions in both the economics and finance literature. In particular, Carroll (1997) emphasized in a seminal paper that precaution –i.e., sensitivity to future income uncertainty– is, quantitatively

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speaking, one of the main motives for savings. This result has been confirmed by many empirical analyses, including the recent study of Mody, Sandri, and Ohnsorge (2012), which reports that "at least two-fifths of the sharp increase in household saving rates between 2007 and 2009 can be attributed to the precautionary savings motive." Precautionary savings are therefore seen as having significant impacts on aggregate wealth accumulation, with major consequences for the economy and business cycles.[1] Moreover, the issue is also very popular in the corporate finance literature in which the precautionary motive is viewed as being a central ingredient to understand the management of liquidity, and more precisely of cash holdings, by corporates.[2]

Despite the sustained interest in precautionary savings, few general results are available. Indeed, a large number of papers have focused on precautionary savings in two-period economies, in which, by construction, saving only occurs once (in the first period) and income is uncertain in a single period (in the second period). Among these papers, one can cite Drèze and Modigliani (1972), Kimball (1990), Bleichrodt and Eeckhoudt (2005), Courbage and Rey (2007), Eeckhoudt and Schlesinger (2008), Kimball and Weil (2009), Bommier, Chassagnon and LeGrand (2012), Jouini, Napp and Nocetti (2013), Nocetti (2016), this list being far from exhaustive. The extension to many periods or to an infinite horizon has rarely been addressed. We are aware of only two contributions where precautionary savings are analytically studied in an infinite horizon setting with a framework that is flexible enough to discuss the role of risk aversion. These are the papers by van der Ploeg (1993) and Weil (1993). Both rely on specific fully parametrized income processes and on closed-form solutions. Besides these two analytical studies, the problem has also been tackled in infinite horizons using numerical techniques –which therefore also imply assuming specific parametric income processes– as in Wang, Wang, and Yang (2016).

In the current paper, we address the question of precautionary savings in an infinite-horizon setting. We consider infinitely long-lived agents who have to take saving-consumption decisions, while being uncertain about their future incomes. The interest rate is deterministic but possibly time-dependent. A key feature of our approach is that we do not rely on a particular specification of the income process, enabling us to consider complex forms of dependence. We prove that whenever the income process is stochastically monotone,[3] higher risk aversion implies greater precautionary savings. Rather intuitively, more risk averse individuals will opt

[1] For example, Challe and Ragot (2016) showed that household precautionary savings are a major channel for the propagation of economic shocks, in particular during recessions.

[2] The importance of the precautionary motive for corporate liquidity policies was for highlighted in empirical studies (Bates, Kahle and Stulz 2009, Campello, Graham and Harvey 2010, Duchin 2010, Duchin, Ozbas and Sensoy 2010), as well as in surveys of corporate managers (Lins, Servaes and Tufano 2010, Campello, Giambona, Graham and Harvey 2011).

[3] Loosely speaking, it means that good news for today's income cannot convey unfavorable news for income in future periods. A formal definition is provided in Section 3.

for larger amounts of savings in presence of income uncertainty. We thus provide a simple and intuitive connection between risk aversion and saving decisions. To our knowledge, this is the first general result that does not rely on an ad-hoc parametric specification of the income process.

The machinery we develop to prove our result relies on the concept of *comonotonicity* and more precisely of *conditional comonotonicity* as introduced by Jouini and Napp (2004). Intuitively, two processes will be conditionally comonotone if future large realizations for one process are accompanied by large realizations for the other process. The first step of the proof involves showing that income, consumption and continuation utilities are comonotone, even if borrowing constraints can be binding at any future date. It then follows that increasing savings generates a transfer of welfare from states with high continuation utility to states with low continuation utility, which diminishes the dispersion of lifetime utilities and therefore corresponds to a risk reduction. A more risk averse individual cares more about risk reduction, by definition, and therefore chooses higher amounts of savings.

A key point for our analysis involves relating risk aversion to the willingness to reduce dispersion in lifetime utilities. This is possible because we focus on preferences that are monotone, a property generally considered to be very appealing when it comes to modeling rationality but sometimes abandoned to gain flexibility while maintaining tractability (further discussion is provided in Section 2). Monotonicity means that a decision maker never takes an action if another available action provides better realizations in all circumstances. Consider for example, a firm which seeks to maximize its intertemporal profit. Monotonicity implies that if the firm has to choose between a risky project and another one that provides greater intertemporal profits in all circumstances, the firm will opt for the latter. While monotonicity is systematically fulfilled when using expected utility specifications, it imposes non trivial restrictions when using recursive representations of preferences (see Bommier, Kochov, and LeGrand 2017). The numerical examples provided in Section 5 actually show that our results do not hold with non-monotone frameworks, therefore underlining the benefit of bringing the assumption of preference monotonicity to the forefront.

2 Risk aversion in intertemporal frameworks

Multidimensional utility theory and risk aversion. The notion of risk aversion was initially introduced for preferences over unidimensional monetary payoffs (Arrow 1963, and Pratt 1964). In such a simple case, when one further assumes the expected utility theory framework, risk aversion is characterized by the concavity of the utility function. Greater concavity implies a stronger degree of risk aversion. Yaari (1969) extended the analysis by allowing for all kinds of payoffs (multi-dimensional, non-monetary, etc). To do so, he introduced a general procedure to compare risk aversion. This procedure states that a decision maker A should be considered as being at least as risk averse as another decision maker B if every risky choice that is acceptable to A (compared with receiving some deterministic reference payoff) is also acceptable to B (compared with the same reference payoff). The procedure is extremely general, since it does not presuppose a specific model of choice under uncertainty and applies to any kind of payoff. Yaari's approach has subsequently become the standard way of comparing risk aversion, and is for example used in Epstein and Zin (1989).

A well known feature of Yaari's procedure is that two decision makers can be compared in terms of risk aversion only if they have the same preferences over deterministic prospects (i.e., riskless payoffs). Several contributions, as those of Richard (1975), Duncan (1977) or Karni (1979), avoid this restriction by quantifying risk aversion through specific risk premiums. Although these approaches are insightful to understand the degree of risk taking of individuals in specific settings, they are not suited to studying the causal relation between risk aversion and savings. Indeed, to draw conclusions on the role of risk aversion, it is key to understand the consequences of an increase in risk aversion, while everything else, including preferences over deterministic prospects, remains unchanged. Otherwise, we could end up with the hardly tenable conclusion that heterogeneity in saving behaviors could be attributed to heterogeneity in risk aversion, even when there is no risk at play. This necessity of focusing on models that are flexible enough to disentangle risk aversion and the ranking of deterministic prospects drastically reduces the set of models that can be used. We review below some of the possibilities.

Expected utility. The standard additive expected utility model assumes that the decision maker maximizes the following expectation:

$$E\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right], \tag{1}$$

where c_t is consumption in period t, β is a constant discount factor and u an instantaneous utility function. In this model, β and u are pinned down (up to a positive affine transformation for u) by preferences over deterministic prospects. This model therefore lacks flexibility to explore the role of risk aversion while keeping preferences over deterministic prospects unchanged.

One solution to study the role of risk aversion within the expected utility framework involves extending expression (1) by assuming that the decision maker maximizes:

$$\psi^{-1} E\left[\psi\left(\sum_{t=0}^{\infty} \beta^t u(c_t)\right)\right], \tag{2}$$

where the function ψ is an increasing function which plays no role in absence of risk. Augmenting its concavity, however, generates an increase in risk aversion in the usual sense of Yaari. This route, initially suggested by Kihlstrom and Mirman (1974), was followed by several authors (Pye 1973, Ahn 1989, van der Ploeg 1993, among others). Such preferences have nevertheless been criticized for being non-stationary, raising time-inconsistency issues if the function ψ is not exponential, and implying time-dependent tastes in the exponential case (see Epstein and Zin 1989, pp. 951-952).

Recursive preferences. To study risk aversion without being subject to the above criticism, Epstein and Zin (1989) extended the recursive approach of Kreps and Porteus (1978) to an infinite-horizon setting. Decision makers maximize at time t a utility function U_t which is linked to the (random) continuation utility U_{t+1} through the following recursion:

$$U_t = u(c_t) + \beta\phi^{-1}\left(E_t\left[\phi(U_{t+1})\right]\right). \tag{3}$$

The function ϕ is an increasing function which has no impact on preferences over deterministic prospects. Its concavity can be directly interpreted in terms of risk aversion, with greater concavity implying greater risk aversion in the sense of Yaari.[4] The recursive construction of equation (3) rules out any possible problem of time-inconsistency or of time-varying tastes. Moreover, the standard additive case of equation (1) is obtained when ϕ is affine. Such recursive preferences have become a standard tool to explore the role of risk aversion.[5]

The most widespread recursive specification –the so-called Epstein-Zin-Weil (EZW, henceforth) specification, which can be found in Epstein and Zin (1989) and Weil (1990)– is obtained when $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ for some parameter $0 < \gamma \neq 1$ and when ϕ exhibits constant relative risk aversion (i.e., $\phi(u) = \frac{((1-\gamma)u)^{\frac{1-\alpha}{1-\gamma}}}{1-\alpha}$ for $0 < \alpha \neq 1$ or $\phi(u) = \frac{1}{1-\gamma}\log((1-\gamma)u)$ for $\alpha = 1$, where α is the parameter controlling for risk aversion). These preferences are popular for

[4] See for example Chew and Epstein (1991) for a formal result.

[5] Another popular way to introduce Kreps-Porteus preferences involves using the following recursion:

$$V_t = u^{-1}\left(u(c_t) + \beta u(\widetilde{\phi}^{-1}(E[\widetilde{\phi}(V_{t+1})]))\right). \tag{4}$$

However, setting $U_t = u(V_t)$ and $\phi = \widetilde{\phi} \circ u^{-1}$ in (4) again yields specification (3). Whether one uses (3) or (4) is thus just a matter of normalization and notation. To avoid confusion, we will systematically rely on the normalization used in (3).

disentangling risk aversion from the intertemporal elasticity of substitution (equal to $\frac{1}{\gamma}$).[6] A well known feature of the EZW specification is that the comparison of consumption programs is unaffected when multiplying all consumption levels by a constant. Wealth effects thus reduce to simple scaling effects and have no impact on the optimal propensity to save. In practice, this significantly simplifies the analysis of problems in which income or asset returns are random, an aspect that contributed to the popularity of the specification.

In his work on precautionary savings, Weil (1993) also uses $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ with $0 < \gamma \neq 1$, but considers $\phi(u) = \frac{1-e^{-\lambda((1-\gamma)u)^{\frac{1}{1-\gamma}}}}{\lambda}$ with $\lambda > 0$. To avoid possible confusion with EZW preferences, we will refer to this specification as *Weil (1993)'s preferences.*[7] Increasing λ generates an increase in risk aversion, but the function ϕ is neither concave nor convex. Such preferences therefore admit no simple comparison with the standard additive model in terms of risk aversion, exhibiting greater risk aversion at some utility levels and lower risk aversion at other utility levels. In particular, the limit case $\lambda = 0$ does not provide the usual additive model.[8]

Risk-sensitive preferences. Although rarely mentioned in the literature, Kreps-Porteus preferences generally fail to fulfill a rather natural property of preferences, which is that of monotonicity. Preference monotonicity stipulates that an agent will not take an action if another available action is preferable in all circumstances. Consider for example a standard two-period consumption-saving problem, where uncertain second period income is distributed over an interval $[y_{min}, y_{max}]$. Monotonicity requires that an agent will not save more than she would do if anticipating second period income y_{min} for sure, or less than she would do if anticipating y_{max} for sure. With non-monotone preferences, this may not be the case. For example, with EZW preferences, a decision maker who learns that she might receive some positive bonus at the end of the year might decide to react to this information by lowering her current consumption and saving more.[9] The agent would react to good news (e.g., the possibility of being awarded a bonus) by taking extra precautions (saving more). One may consider this behavior to be unintuitive and contrary to the notion of precaution. This may

[6] EZW preferences can also accommodate the case of an intertemporal elasticity of substitution equal to one. This corresponds to $u(c) = \log(c)$ and a function ϕ which –because of utility renormalization– exhibits constant absolute (not relative) risk aversion. Such a specification, used for example in Tallarini (2000), also fits into the class of risk-sensitive preferences that we discuss below.

[7] In his paper, Weil uses the normalization associated with equation (4) rather than the one associated with equation (3). The function $\widetilde{\phi}$ is then exponential, but this is different from the risk-sensitive preferences described below which assume ϕ (and not $\widetilde{\phi}$) to be exponential. Weil (1993)'s preferences can accommodate the case $\gamma = 1$ by setting $u(c) = \log(c)$ and $\phi(u) = \frac{1-\exp(-\lambda \exp(u))}{\lambda}$.

[8] For instance, in a two-period setup, the limit case $\lambda = 0$ provides $\frac{c_0^{1-\gamma}}{1-\gamma} + \frac{\beta}{1-\gamma}\left(E\left[c_1\right]\right)^{1-\gamma}$, which differs from the standard additive utility function $\frac{c_0^{1-\gamma}}{1-\gamma} + \beta\frac{E\left[c_1^{1-\gamma}\right]}{1-\gamma}$ because of Jensen inequality.

[9] See Bommier, Kochov, and LeGrand (2017) for a numerical example.

eventually result in counter-intuitive conclusions about the role of risk aversion.

The only way to combine the recursive approach (3) with the property of monotonicity involves using exponential (or affine) functions ϕ (see Bommier, Kochov, and LeGrand, 2017 for a formal representation result).[10] With $\phi(u) = \frac{1-e^{-ku}}{k}$, equation (3) can be rewritten as:

$$U_t = \begin{cases} u(c_t) - \frac{\beta}{k} \log\left(E_t[e^{-kU_{t+1}}]\right) & \text{, if } k \neq 0, \\ u(c_t) + \beta E_t[U_{t+1}] & \text{, if } k = 0. \end{cases} \tag{5}$$

Consistently with the role of ϕ described above, the parameter k only plays a role in presence of uncertainty and governs risk aversion, with greater risk aversion being related to larger values of k. The case $k = 0$ in equation (5) corresponds to the standard additively separable expected utility model. Intuitively, these preferences are monotone because the instantaneous utility can be "entered" into the certainty equivalent: $U_t = -\frac{\beta}{k} \log\left(E_t[e^{-\frac{k}{\beta}(u(c_t)+\beta U_{t+1})}]\right)$ if $k \neq 0$ and $U_t = E_t[u(c_t) + \beta U_{t+1}]$ if $k = 0$. This is impossible with other forms of certainty equivalents.

Recursive preferences as in equation (5) are usually called risk-sensitive preferences and were first introduced by Hansen and Sargent (1995) in a work inspired by control theory. As explained in Hansen, Sargent, Turmuhambetova, and Williams (2006, p. 55), the risk-sensitive decision rule can be formalized as a Nash equilibrium solution of a game where "a maximizing player ('the decision maker') chooses a best response to a malevolent player ('nature') who can alter the stochastic process within prescribed limits". With such a game-theoretic interpretation, the counterpart of the assumption of preference monotonicity is the elimination of dominated strategies. The monotonicity of risk-sensitive preferences is simply due to the fact that the Nash equilibrium rules out the choice of dominated strategies.

It is worth mentioning that risk-sensitive preferences do not impose restrictions on the intertemporal elasticity of substitution, which does not need to be constant. Moreover, with risk-sensitive preferences, wealth effects have typically non-trivial consequences (the exception being when a constant intertemporal elasticity of substitution of one is assumed) which generally precludes the derivation of closed form solutions. However, as we will see, the fact that they are monotone allows us to derive new results on the role of risk aversion.

3 Precautionary savings in dynamic settings

To further pursue our analysis, we focus on risk-sensitive preferences, which are the only ones to be recursive and monotone, while being flexible enough to disentangle risk aversion from

[10] In particular EZW preferences are monotone only if intertemporal elasticity of substitution is equal to one or if they correspond to the standard additive model. Weil (1993)'s preferences are non-monotone in all cases.

intertemporal substitution. More specifically, we consider infinitely long-lived agents with preferences given by recursion (5). Agents only differ by the risk aversion parameter $k \geq 0$.[11] The function u is assumed to be increasing, concave, and twice continuously differentiable.

Agents are endowed with an exogenous income process $(y_t)_{t\geq 0}$ which does not need to be specified to derive our results. We only assume that this process is *stochastically monotone* meaning that for all $t \geq 0$ and $x \in \mathbb{R}$ the function $(y_0, y_1, \ldots, y_t) \mapsto \text{Prob}(y_{t+1} \geq x | y_0, y_1, \ldots, y_t)$ is non-decreasing. The assumption of stochastic monotonicity formalizes the idea that a good income realization at a given date cannot convey negative information for subsequent periods. Most income processes used in the literature comply with such an assumption. This is for example the case of standard autoregressive processes, as with those considered for instance in van der Ploeg (1993) and Weil (1993). Technically speaking, the assumption of a stochastically monotone income process does not rule out extremely rapid income growth or income decline, which could result in convergence problems, with no existing solution to recursion (5). Rather than introducing a set of technical assumptions, we simply assume that the income process and the preference parameters are such that convergence problems do not occur.

We consider the saving decision at time t of agents with wealth w and realized income trajectory denoted by $y^t = (y_0, \ldots, y_t)$.[12] The gross interest rate between dates t and $t+1$, denoted by R_{t+1}, is time-varying but deterministic. Let $w \mapsto V_t^k(w, y^t)$ be the indirect utility at time t of the agent with risk aversion k. We have:

$$V_t^k(w, y^t) = \max_{s_t \in \mathbb{R}} \; u(c_t) - \frac{\beta}{k} \log E_t \left[e^{-k V_{t+1}^k (R_{t+1} s_t, y^{t+1})} \right], \tag{6}$$
$$\text{s.t. (i) } y_t + w - s_t = c_t, \text{ (ii) } c_t > 0, \text{ and (iii) } s_t \geq \underline{s}_t(y^t),$$

where $\underline{s}_t(y^t)$ is the borrowing constraint at time t. This limit may reflect "natural" constraints –also sometimes called solvency constraints– due to the fact that an agent cannot borrow more than what she can repay in the worst scenario, or may be exogenous and related to market imperfections. Since the income process is not stationary, the borrowing constraint is likely to depend on the information obtained through past income realization information. This is why $\underline{s}_t$ is explicitly set as a function of y^t, the income trajectory. In line with the assumption of a stochastically monotone income process, we assume that $\underline{s}_t(y^t)$ is non-increasing in y^t. Intuitively, this means that good news concerning income cannot convey negative information

[11] By imposing $k \geq 0$, we restrict the study to agents that are at least as risk averse as a standard additively separable expected utility maximizer. Cases where $k < 0$ would be be difficult to address, due to potential non-convexity issues.

[12] We indicate the whole income history, as the income process is not necessarily Markovian. Expectations regarding future income may then depend on the whole income history.

about future borrowing constraints. This specification of course includes the case where borrowing constraints are zero and independent of past income ($\underline{s}_t = 0$ for all t), which is a frequent assumption in the literature. Finally, we assume that for all t and all income histories y^t at date t, and all income realizations y_{t+1} in period $t+1$, we have $R_t \underline{s}_t(y^t) + y_{t+1} > \underline{s}_{t+1}(y^{t+1})$, meaning that saving $\underline{s}_t(y^t)$ in period t is possible.[13]

If $w \leq \underline{s}_t(y^t) - y_t$, the above program is not well defined since no saving and consumption level fulfill the constraints (i) to (iii) in problem (6). For any $w > \underline{s}_t(y^t) - y_t$, we denote by $s_t^k(w, y^t)$ the solution of the optimization problem, that is the amount that the agent with wealth w and risk aversion k chooses to save. We can now state the following result:

Proposition 1 (Propensity to save) *Greater risk aversion implies a higher propensity to save at any date. Formally, for all $t \geq 0$, y^t, $w > \underline{s}(y^t) - y_t$, k, $k' \geq 0$, the following implication holds:*

$$k' \geq k \quad \Rightarrow \quad s_t^{k'}(w, y^t) \geq s_t^k(w, y^t).$$

Proof sketch. The formal proof is provided in Appendix. The main intuition of the proof can be summarized as follows. We first establish that, even in the presence of binding borrowing constraints, consumption, income and continuation utility fulfill a property of conditional comonotonicity as introduced by Jouini and Napp (2004). This means that, for a given history y^t, high income realizations at time $t+1$ will also correspond to high continuation utilities and high consumption levels. The latter aspect implies that the impact on continuation utility of a marginal increase in saving s_t is larger if information at date $t+1$ reveals a low income rather than a high income. Thus, an increase in savings generates a transfer of welfare from states with high continuation utility to states with low continuation utility, achieving a risk reduction. Highly risk averse agents value this risk reduction more than low risk averse agents and therefore end up saving more. ■

Proposition 1 shows that the more risk averse agent has a greater propensity to save than the less risk averse one. The comparison is established for agents having the same amount of wealth in period t. In dynamic problems, however, agents with different degrees of risk aversion will hold different amounts of wealth at time t. The difference in their saving behaviors will therefore result both from differences in propensities to save and from differences in accumulated wealth levels. The function $w \mapsto s_t^k(w, y^t)$ being non-decreasing (i.e., wealthier agents save more), both effects go in the same direction, providing the following result:

[13] Not assuming such an inequality would necessarily lead to introducing another borrowing constraint preventing the agent from making a decision likely to yield an infeasible situation in the following period. Our formalization assumes that such constraints are already reflected in $\underline{s}_t(y^t)$.

Corollary 1 (Wealth accumulation) *Greater risk aversion implies greater accumulated wealth. Formally, denote by* $(w_t^k)_{t\geq 0}$ *the wealth process that results from the optimal saving behavior of an agent with risk aversion* k.[14] *The following implications holds:*

$$k' \geq k \quad \Rightarrow \quad \forall t \geq 0,\ w_t^{k'} \geq w_t^k.$$

Corollary 1 indicates that if we compare two agents receiving the same exogenous stochastic income and facing no other risks, the more risk averse agent will be wealthier than the less risk averse agent, at all ages and in all circumstances.

The notion of precautionary savings refers to the amount of wealth accumulated to cope with income uncertainty. Formally, precautionary savings at time t are usually defined as the difference between the wealth accumulated when income risk in not insurable and the wealth accumulated when income is fully insured. In our framework, which allows for stochastic borrowing constraints, the definition of precautionary savings requires some assumptions be made regarding the path of deterministic borrowing constraints when income can be fully insured. Those assumptions, however, have no effect on our result regarding the relation between risk aversion and precautionary savings. Suppose for example that when income risk can be fully insured, the agent faces a sequence of deterministic income $\widehat{y}_t$ and of deterministic borrowing constraints $\widehat{\underline{s}}_t$. Maximization of the utility defined in (5) generates a wealth trajectory $\widehat{w}_t$, which is independent of k, since there is no risk at play. Following the literature, we define *precautionary savings* as:

$$\Delta w_t^k = w_t^k - \widehat{w}_t.$$

The wealth trajectory $\widehat{w}_t$, and therefore precautionary savings Δw_t^k, may of course depend on the specification of the sequence $\hat{\underline{s}}_t$. However, since $\widehat{w}_t$ is independent of k, it immediately follows from Corollary 1 that:

Corollary 2 (Precautionary savings) *Greater risk aversion implies larger precautionary savings. Formally:*

$$k' \geq k \quad \Rightarrow \quad \forall t \geq 0,\ \Delta w_t^{k'} \geq \Delta w_t^k.$$

A distinct but related question is whether an agent with risk aversion k is *prudent*, i.e., whether $\Delta w_t^k \geq 0$ for all $t \geq 0$. A partial answer is obtained when combining results that are known to hold for the additive specification (i.e., when $k = 0$) and those of the current paper. Indeed, Light (2016), who extends previous results of Miller (1976) and Kimball (1990), shows that with the additive specification, when the income process is Markovian

[14] Formally speaking, w_t^k is defined by $w_0^k = 0$ and $w_{t+1}^k = R_{t+1} s_t^k(w_t^k, y^t)$ for $t \geq 0$.

and the borrowing constraints are given by $\underline{s}_t = \hat{\underline{s}}_t = 0$, a sufficient condition to have non-negative precautionary savings is the positivity of the third derivative of the instantaneous utility function ($u''' \geq 0$).[15] It then follows from Corollary 2 that Light's conclusion extends to any risk-sensitive preferences with $k \geq 0$. Note however that the current paper's setting is more general than Light's, since it allows for non-Markovian income processes and stochastic borrowing constraints. What determines the sign of precautionary savings in such a general setting remains an open question, which looks challenging even when $k = 0$.

4 Relation to the previous literature

As mentioned in the introduction, most of the theoretical literature was developed in two-period frameworks. However, a two-period economy can be modeled as an infinite period economy, in which income and consumption remain constant forever from the second period on. Thus, our result that risk aversion increases precautionary savings also applies to two-period settings, and can therefore be compared with the conclusions of Kimball and Weil (2009) and Bommier, Chassagnon and LeGrand (2012) who discuss the role of risk aversion in two-period models. Kimball and Weil focus on Kreps-Porteus preferences, but do not impose preference monotonicity. They find that an increase in risk aversion may increase or decrease precautionary savings (see Proposition 7 in Kimball and Weil, 2009). Interestingly, Kimball and Weil obtain this ambiguous conclusion by considering non-monotone EZW specifications. We thus readily know that our results of Section 3 do not extend to EZW preferences. In fact, the contrast between the ambiguous findings of Kimball and Weil (2009) and the unambiguous statements of Proposition 1 and Corollaries 1 and 2 highlights the significance of assuming preference monotonicity when investigating the link between risk aversion and precautionary savings. A way to formally connect our results with those of Kimball and Weil (2009) involves using the formula (16) of their paper to derive that the coefficient of relative prudence associated with risk-sensitive preferences is $2kcu'(c) - \frac{cu'''(c)}{u''(c)}$. This coefficient, which is informative about precautionary savings in two-period settings with infinitesimally small risks, is increasing in k, and positive as soon as $u''' > 0$ and $k \geq 0$. This is consistent with the results discussed in the previous section.[16]

Bommier, Chassagnon and LeGrand (2012) focus on preferences that are monotone, but

[15]When $k = 0$, the property $u''' \geq 0$ can also be related to aversion to downward risk, when considering risk bearing on consumption at a single period of time, holding consumption in other periods constant. See, e.g., Menezes, Geiss and Tressler (1980), Chiu (2005), Denuit and Eeckhoudt (2010), or Ebert and Wiesen (2011).

[16]Note however that one cannot use the coefficient of relative prudence to infer conclusions on what occurs with large risks. For example, with EZW specification, the coefficient of relative prudence is given by $\alpha(1 + \frac{1}{\gamma})$, thus increasing with risk aversion (α), while –as we mentioned above– Kimball and Weil (2009) proved that risk aversion may negatively impact precautionary savings when risks are large.

not necessarily recursive. Using dominance arguments, they prove that risk aversion increases precautionary savings, in line with our current analysis. However, extending their approach to an infinite horizon would involve considering non-recursive preferences and would raise time-consistency issues.

Meanwhile, analyses in infinite-horizon settings are much rarer with only two theoretical contributions, van der Ploeg (1993) and Weil (1993), that investigate the role of risk aversion. As previously mentioned, van der Ploeg (1993) uses the expected utility representation (2) with $\psi(u) = \frac{1-e^{-ku}}{k}$, while Weil (1993) relies on the recursive representation (3) with $\phi(u) = \frac{1-e^{-\lambda((1-\gamma)u)^{\frac{1}{1-\gamma}}}}{\lambda}$. Furthermore, van der Ploeg (1993) assumes a quadratic instantaneous utility function u, while Weil (1993) assumes that $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$. Both articles consider particular income processes to derive closed form solutions. One may naturally wonder whether their results could be extended to more general stochastic processes.

To start with, remark that when $\psi(u) = \frac{1-e^{-ku}}{k}$, the expected utility preferences represented by (2) also admit a recursive representation provided by:

$$U_t = \begin{cases} u(c_t) - \frac{\beta}{k_t} \log\left(E_t[e^{-k_t U_{t+1}}]\right) & , \text{ if } k \neq 0, \\ u(c_t) + \beta E_t[U_{t+1}] & , \text{ if } k = 0. \end{cases} \tag{7}$$

where $k_t = k\beta^t$. Recursion (7) is very similar to the recursion (5) that defines risk-sensitive preferences, the difference being that (7) assumes a degree of risk aversion k_t that depends on t. This dependence in t generates the "changing taste" feature underlined by Epstein and Zin (1989).[17] In technical terms, dealing with recursion (7) instead of recursion (5) has almost no impact for our proof (one just needs to change k by $k\beta^t$ in all equations). The results of Proposition 1 and Corollaries 1 and 2 thus extend to the preferences used in van der Ploeg (1993), with in fact no need to restrict the analysis to quadratic instantaneous utilities or specific income processes.

The preferences used by Weil (1993) are non-monotone and our proof strategy does not apply to his setting. In fact, with Weil (1993)'s preferences the positive relation between risk aversion and precautionary savings is obtained for some specific stochastically monotone income processes (such as those considered in Weil's paper) but does not hold for others. The following section provides an example where the relation does not hold.

[17]Note that this problem of changing taste disappears if one sets $\beta = 1$, providing the multiplicative model axiomatized in Bommier (2013). However, following such a route requires the infinite-horizon setting be abandoned in order to avoid convergence issues.

5 Numerical example

In this section, we present a numerical application that illustrates our main result and the importance of the monotonicity property. We will indeed contrast the results obtained with risk-sensitive, EZW and Weil (1993)'s preferences. All three cases will rely on the same income process and on the same preferences over deterministic prospects.

Preference parameters			
	γ	β	
	2	0.96	
Income parameters at dates 0 and 1			
y_0	$y_{1,l}$	$y_{1,h}$	
20.0	0.1	0.5	
Income parameters at dates $t \geq 2$			
θ_h	θ_l	ρ	σ
20.0	10.0	0.95	0.1%

Table 1: Calibration

Income process. An agent is endowed with a certain income $y_0 > 0$ at date 0. At date 1, there are two possible states labeled h and l, that each occurs with probability 1/2. The state realization at date 1 fully determines the income in period 1 as well as the level of the income process in the following periods. More precisely, at date 1 and in state $\kappa = h, l$, the agent receives income $y_{1,\kappa}$ with $y_{1,h} \geq y_{1,l} > 0$. The income at any future date $t \geq 2$ is equal to $y_{t,\kappa} = \theta_\kappa e^{z_t}$, where θ_κ is a scaling parameter with $\theta_h > \theta_l$, and $(z_t)_{t\geq 1}$ is an AR(1) process driving income uncertainty after date 2. Formally, $z_1 = 0$ and for all $t \geq 2$, $z_t = \rho z_{t-1} + \sigma\varepsilon_t$, with $\varepsilon_t \overset{\text{iid}}{\sim} \mathcal{N}(0,1)$ and ρ, $\sigma \geq 0$.[18] The uncertainty process $(z_t)_{t\geq 2}$ is independent of the state $\kappa = h, l$ and stochastically monotone. Moreover, since $\theta_h > \theta_l$, $(y_{t,h})_{t\geq 2}$ dominates at the first order $(y_{t,l})_{t\geq 2}$, implying that the income process $(y_t)_{t\geq 0}$ is stochastically monotone. Finally, agents are prevented from borrowing: $\underline{s}_t = 0$ at all dates t.

Preferences. We consider recursive preferences, as in Section 2. We compare risk-sensitive preferences, EZW preferences, and those of Weil (1993). The instantaneous utility function u and the time-discount parameter β are the same for the three specifications with $u(c) = (1-\beta)\frac{c^{1-\gamma}}{1-\gamma}$ for some constant $0 < \gamma \neq 1$. The only difference comes from the risk aversion

[18] Note that the initial value z_1, set to 0, does not impact the agents' income in period 1, but only serves to initialize the process z_t.

function ϕ that enters recursion (3). As already mentioned, for risk-sensitive preferences, we consider $\phi(x) = \frac{1-e^{-kx}}{k}$, with $k \geq 0$. For EZW specification, we take $\phi(u) = \frac{((1-\gamma)u)^{\frac{1-\alpha}{1-\gamma}}}{1-\alpha}$, with $\alpha \geq \gamma$, while Weil (1993)'s specification corresponds to $\phi(u) = \frac{1-e^{-\lambda((1-\gamma)u)^{\frac{1}{1-\gamma}}}}{\lambda}$, with $\lambda \geq 0$.

Calibration and numerical resolution. Our calibration is summarized in Table 1. The gross interest rate is assumed to be constant and equal to $R = 1$.

Let us now describe our numerical solution method. We first need to solve the consumption-saving problem at date 2 in both states h and l. The problem is similar in both states and is a standard consumption-saving problem with stochastic income. The state-space is two-dimensional and consists of the current value of the process z and the agent's savings. We discretize the autoregressive process $(z_t)_{t\geq 1}$ as a finite Markov chain using the Tauchen (1986) method, with seven nodes and a multiple of the standard deviation equal to 3. We use a grid with 20 non-evenly spaced points for savings. We evaluate the continuation utility at values out of the grid using a simple linear interpolation. We then compute optimal policy and value functions in states h and l using a policy function iteration algorithm. Once the problem is solved for date 2, we deduce the optimal savings at date 0 by backward induction.[19]

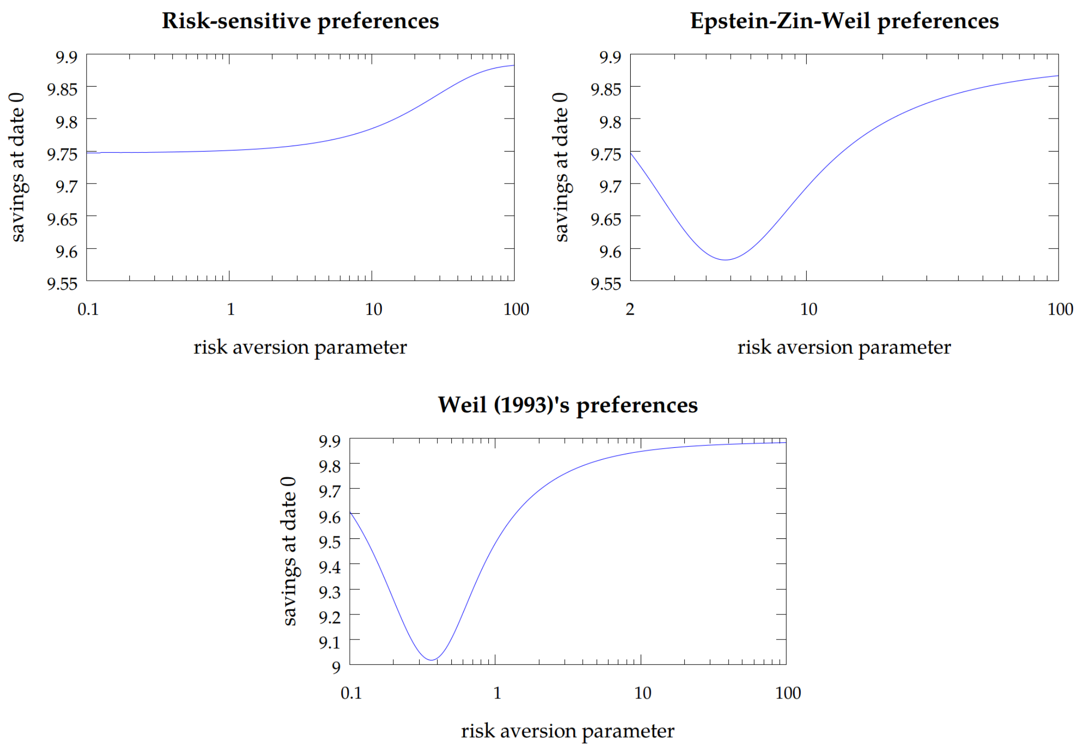

Figure 1: Impact of the risk aversion parameter on savings

[19]Fortran codes for solving the program and reproducing the figures, as well as the related documentation, are available at `http://francois-le-grand.com/docs/codes/BL_PrecautionarySavings_codes.zip`.

Results. In Figure 1, we plot the optimal savings at date 0 as a function of the risk aversion parameter, which is k for risk-sensitive preferences, α for EZW preferences, and λ for Weil (1993)'s preferences (with a logarithmic scale on the x-axis). Note that risk-sensitive preferences with $k = 0$ and the EZW specification with $\alpha = \gamma$ provide the same amount of savings, both models reducing then to the standard additive specification (1). However, as noted in Section 2, the limit case $\lambda = 0$ of Weil (1993)'s preferences is different, and logically yields another amount of savings.

With risk-sensitive preferences, savings are increasing with the risk aversion parameter k. This illustrates our result of Proposition 1. However, there is no such a monotonic relationship with EZW or Weil (1993)'s preferences. This example is an illustration that in absence of monotonicity, Proposition 1 no longer holds.

6 Concluding remarks

The current paper addresses the question of precautionary savings, which is one of the most studied issues in the theory of choice under uncertainty. We consider a general approach, where uncertainty is not assumed to take any specific form, abandoning by the same token any hope of deriving closed-form solutions. Despite this, we were able to prove the existence of a positive relation between risk aversion and savings. Our approach emphasizes the role of two fundamental assumptions: stochastic monotonicity of the income process and preference monotonicity.

Intuitively, under the assumption of preference monotonicity, risk aversion can be seen as the willingness to redistribute utility from good states of the world to bad states of the world, or in other words, to opt for a strategy "closer" to the ones that would be chosen in bad states. Moreover, due to the assumption of a stochastically monotone income process, low future income realizations (which would be associated with greater savings levels in case of perfect foresight) can be unambiguously considered as bad states. The combination of both aspects leads to finding that risk aversion increases precautionary savings. This seems to be a fairly intuitive result. However, for a number of reasons, such as the focus on the additively separable expected utility model, the use of non-monotone preferences, or the difficulty in deriving closed-form solutions, this fundamental relation between risk aversion and savings has hitherto largely remained unclarified. This paper helps shed light on this relation, which might be key to understanding heterogeneity in saving behaviors.

Appendix

A Proof of Proposition 1

A.1 Mathematical preamble

All processes are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, endowed with the filtration $(\mathcal{F}_t)_{t\geq 0}$. For any random variable X, we denote its expectation, under $\mathbb{P}$, by $E[X] = \int_\Omega X(\omega)d\mathbb{P}(\omega)$, and its variance, under $\mathbb{P}$, $V[X] = E[X^2] - E[X]^2$ –when they exist. The covariance under $\mathbb{P}$ between two random variables X and Y is denoted by $cov(X, Y) = E[XY] - E[X]E[Y]$, when it exists. Let L be an a $\mathbb{P}-$almost surely non-negative random variable such that $E[L] = 1$. The function $\mathbb{Q} : \mathcal{F} \to \mathbb{R}$, defined for any $A \in \mathcal{F}$ by $\mathbb{Q}(A) = \int_{\omega\in A} L(\omega)d\mathbb{P}(\omega)$ is a probability measure. The expectation under $\mathbb{Q}$ denoted by $E^{\mathbb{Q}}[\cdot]$, when it exists, verifies for any random variable X: $E^{\mathbb{Q}}[X] = E[LX]$. We also define (when they exist) the variance and the covariance under $\mathbb{Q}$: $V^{\mathbb{Q}}[X] = E[LX^2]-E[LX]^2$ and $cov^{\mathbb{Q}}(X, Y) = E[LXY]-E[LX]E[LY]$. The notations are straightforward to extend for conditional moments.

The proof will utilize a property of conditional comonotonicity and some significant related results. Our presentation, below, is taken from Jouini and Napp (2004). Let $\mathcal{G}$ be a sub-sigma algebra of $\mathcal{F}$. We start by characterizing the comonotonicity conditionally to $\mathcal{G}$.

Definition 1 (Comonotonicity conditional to $\mathcal{G}$) *Two random variables X and Y defined on $(\Omega, \mathcal{F}, \mathbb{P})$ are said to be comonotonic conditionally to $\mathcal{G}$ if and only if there exist a random variable ξ and two functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ such that: (i) $f(., x)$ and $g(., x)$ are $\mathcal{G}$-measurable; (ii) $f(\omega, .)$ and $g(\omega, .)$ are non-decreasing; and (iii) $(X, Y) = (f(\omega, \xi), g(\omega, \xi))$ almost surely.*

The intuition of the above definition is straightforward: X and Y are comonotonic conditionally to $\mathcal{G}$ if their "restrictions" to $\mathcal{G}$ are comonotononic. We now turn to the definition of conditional comonotonicity for processes.

Definition 2 (Conditional comonotonicity) *Two adapted random processes (X_t) and (Z_t) defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ are said to be conditionally comonotonic if, for $t \geq 0$, the random variables X_{t+1} and Z_{t+1} are comonotonic conditionally to $\mathcal{F}_t$.*

Conditional comonotonicity generalizes the concept of comonotonicity to stochastic processes. We conclude this section with a useful result:

Proposition 2 (Jouini and Napp, 2004) *Let X and Y be two random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. If X and Y are comonotonic conditionally to $\mathcal{G}$, then $cov^{\mathbb{Q}}_{\mathcal{G}}(X, Y) \geq 0$ for all probability measures $\mathbb{Q}$ absolutely continuous with respect to $\mathbb{P}$.*

In the remainder, we will say that two random variables X and Y are *anticomonotonic* when X and $-Y$ are comonotonic.

A.2 Comonotonicity properties

As a first step in the proof of Proposition 1, we establish some comonotonicity properties that relate income, consumption and continuation utilities.

The program (6) of the agent can be expressed as follows:

$$V_t^k(w_t, y^t) = \max_{\underline{s}_t \leq s_t < y_t + w_t} u(y_t - s_t + w_t) - \frac{\beta}{k} \log E_t \left[e^{-kV_{t+1}^k(R_{t+1}s_t, y^{t+1})} \right], \tag{8}$$
$$\text{s.t. } w_{t+1} = R_{t+1}s_t,$$

where $E_t[\cdot]$ is the expectation conditional on $\mathcal{F}_t$. We define W_t^k by:

$$W_t^k(w_t^k, y^t) = -kV_t^k(w_t^k, y^t) = \min_{\underline{s}_t \leq s_t < y_t + w_t} -ku(y_t - s_t + w_t^k) + \beta \log E_t e^{W_{t+1}^k(R_{t+1}s_t, y^{t+1})}. \tag{9}$$

To derive our result, we need to differentiate the value function. Following standard arguments of dynamic programming, we can show that since u is increasing and concave, so is the value function. Moreover, since u is assumed to be continuously derivable, so is the value function, as proved by Benveniste and Sheinkman (1979).[20] We will therefore denote by $W_{t,w}^k$ and $W_{t,k}^k$ the derivatives of W_t^k with respect to w_t and k respectively. We respectively denote by c_t^k and s_t^k the optimal consumption and savings of an agent maximizing (8) endowed with the risk aversion parameter k. The strict concavity of u guarantees the uniqueness of s_t^k. The first-order condition provides:

$$ku'(c_t^k) \geq -\beta R_{t+1} \frac{E_t \left[W_{t+1,w}^k(w_{t+1}^k, y^{t+1}) e^{W_{t+1}^k(w_{t+1}^k, y^{t+1})} \right]}{E_t \left[e^{W_{t+1}^k(w_{t+1}^k, y^{t+1})} \right]}, \tag{10}$$

where equality holds if $s_t^k > \underline{s}_t$. The envelop theorem yields the following equalities (which are valid whether the constraint $s_t^k \geq \underline{s}_t$ is binding or not):

$$W_{t,w}^k(w_t^k, y^t) = -ku'(y_t - s_t^k + w_t^k) = -ku'(c_t^k), \tag{11}$$

$$W_{t,k}^k(w_t^k, y^t) = -u(c_t^k) + \beta E_t \left[W_{t+1,k}^k \frac{e^{W_{t+1}^k(w_{t+1}^k, y^{t+1})}}{E_t \left[e^{W_{t+1}^k(w_{t+1}^k, y^{t+1})} \right]} \right]. \tag{12}$$

[20]Stokey and Lucas (1989, Theorem 4.11) also provide a proof of the differentiability of the value function in a slightly more general framework.

Note that as a corollary of the continuous differentiability of the value function, the consumption level and the saving choices are continuous in k. We now state the following lemma.

Lemma 1 (Comonotonicity of income, consumption and continuation utility) *In the setup of Proposition 1, at any date $t \geq 0$, the optimal consumption process $(c_t^k)_{t\geq 0}$, the income process $(y_t)_{t\geq 0}$ and the continuation utility process $(V_t^k)_{t\geq 0}$ are conditionally comonotonic.*

Proof. We prove the result assuming that there is a finite $T \in \mathbb{N}$ such that income in periods after T is deterministic. Since T can be any finite integer, and $\beta < 1$, a continuity argument implies that the result extends to $T = \infty$.

We prove by reverse induction on t that (i) c_t^k and y_t are comonotonic conditionally to $\mathcal{F}_{t-1}$ and that (ii) y_t and W_t^k are anticomonotonic conditionally to $\mathcal{F}_{t-1}$. At date $t = T$, there is no remaining uncertainty. Conditionally to the filtration $\mathcal{F}_{T-1}$, and in particular for a given history of income $(y_t)_{0\leq t\leq T-1}$ up to date $T-1$, let us consider two realizations y_T and y'_T of y_T (at date T) such that $y_T > y'_T$. Due to the stochastic monotonicity, we know that $y_{T+\tau} \geq y'_{T+\tau}$ for any $\tau > 0$. Credit constraints being non-increasing in past income realizations, the (intertemporal) budget set when receiving y_T is therefore larger than the one obtained when receiving y'_T. The consumption profile $(c_{T+\tau})_{\tau\geq 0}$, chosen when receiving y_T is thus revealed preferred to the consumption profile $(c'_{T+\tau})_{\tau\geq 0}$ chosen when receiving y'_T. Assume that $c_T < c'_T$. Since $(c_{T+\tau})_{\tau\geq 0}$ is revealed preferred to $(c'_{T+\tau})_{\tau\geq 0}$, there must be at least one τ for which $c_{T+\tau} > c'_{T+\tau}$. Moreover since the borrowing limit $\underline{s}_t(y^t)$ is non-increasing in y^t, the borrowing constraint at time T cannot be binding when choosing c_T. We know that $u'(c'_T) \geq \beta^\tau \left(\prod_{k=1}^{\tau} R_{T+k}\right) u'(c'_{T+\tau})$, otherwise it would be optimal to decrease consumption c'_T by a small amount to increase $c'_{T+\tau}$. Since $c_T \leq c'_T$ implies $u'(c_T) \geq u'(c'_T)$ and $c_{T+\tau} > c'_{T+\tau}$ implies that $u'(c'_{T+\tau}) > u'(c_{T+\tau})$, we obtain $u'(c_T) > \beta^\tau \left(\prod_{k=1}^{\tau} R_{T+k}\right) u'(c_{T+\tau})$, contradicting the optimality of $(c_{T+\tau})_{\tau\geq 0}$. We deduce that $c_T \geq c'_T$. We conclude that c_T^k and y_T are comonotonic conditionally to $\mathcal{F}_{T-1}$. Since $W_T^k = -k\sum_{\tau=0}^{\infty} \beta^\tau u(c_{t+\tau}^k)$ (no uncertainty is left after T) and u increasing, we can also conclude that y_T and W_T^k are anticomonotonic conditionally to $\mathcal{F}_{T-1}$, and thus that y_T and V_T^k are comonotonic.

We have shown that points (i) and (ii) hold for $t = T$. We now proceed by induction showing that if they hold for $0 < t \leq T$, they also hold for $t-1$. When the borrowing constraint does not bind at time $t-1$, the Euler equation (10) together with (11) implies:

$$u'(c_{t-1}^k) = (\beta R_t) E_{t-1}\left[u'(c_t^k)\frac{e^{W_t^k(w_t^k, y^t)}}{E_{t-1}[e^{W_t^k(w_t^k, y^t)}]}\right]. \tag{13}$$

Using the induction hypothesis, we know that $u'(c_t^k)\frac{e^{W_t^k(w_t^k, y^t)}}{E_{t-1}[e^{W_t^k(w_t^k, y^t)}]}$ and y_t are anticomonotonic conditionally to $\mathcal{F}_{t-1}$. Since the income process is stochastically monotone, we deduce that

$u'(c_{t-1}^k)$ is non-increasing with y_{t-1}, meaning that c_{t-1}^k and y_{t-1} are comonotonic conditionally to $\mathcal{F}_{t-2}$. Assumptions on the income process and borrowing constraints, and the definition (9) of W_{t-1}^k, imply that W_{t-1}^k is decreasing with y_{t-1}. The induction hypothesis and the comonotonicity of c_{t-1}^k and y_{t-1} conditionally to $\mathcal{F}_{t-1}$ allow us to conclude that W_{t-1}^k and y_{t-1} are anticomonotonic conditionally to $\mathcal{F}_{t-2}$, and thus that y_{t-1} and V_{t-1}^k are comonotonic.

When borrowing constraints bind, denote by $\underline{y}_{t-1}$ the cut-off value of y_{t-1} below which the Euler equation does not hold. For any realization of y_{t-1} below $\underline{y}_{t-1}$, the borrowing constraint at date $t-1$ binds: c_{t-1}^k varies exactly as y_{t-1}. Therefore, c_{t-1}^k, y_{t-1}, and V_T^k are also comonotonic conditionally to $\mathcal{F}_{t-2}$. ■

A.3 Increasing risk aversion

To complete the proof of Proposition 1, we consider the impact of an increase in risk aversion. Ideally, we would wish to differentiate the Euler equation with respect to k and show that $\frac{\partial s_t^k}{\partial k} > 0$. However, this supposes that $k \mapsto s_t^k$ is differentiable, or equivalently that $w \mapsto W_t^k(w, y^t)$ is twice differentiable. Unfortunately, this is not trivial to prove in our setup and standard arguments as in Araujo (1991) or in Santos (1991) do not apply. We will therefore use a method that avoids differentiation of the savings function.

Consider two agents endowed with risk aversion parameters $k' > k$. We distinguish two cases, depending on whether the Euler equation for k holds with equality or not.

First case: Euler equations (10) and (13) hold with equality for agent k. Euler equation for agent k' may not hold with equality. Dropping the dependence in y^t, we deduce from Euler equation (10) for k and k', after some rearrangements, that:

$$\begin{aligned}\frac{k'(u'(c_t^{k'}) - u'(c_t^k))}{\beta R_{t+1}} + \frac{E_t\left[W_{t+1,w}^{k'}(w_{t+1}^{k'})e^{W_{t+1}^{k'}(w_{t+1}^{k'})}\right]}{E_t\left[e^{W_{t+1}^{k'}(w_{t+1}^{k'})}\right]} - \frac{E_t\left[W_{t+1,w}^{k'}(w_{t+1}^{k})e^{W_{t+1}^{k'}(w_{t+1}^{k})}\right]}{E_t\left[e^{W_{t+1}^{k'}(w_{t+1}^{k})}\right]} \geq \\ \frac{(k-k')u'(c_t^k)}{\beta R_{t+1}} - \frac{E_t\left[W_{t+1,w}^{k'}(w_{t+1}^{k})e^{W_{t+1}^{k'}(w_{t+1}^{k})}\right]}{E_t\left[e^{W_{t+1}^{k'}(w_{t+1}^{k})}\right]} + \frac{E_t\left[W_{t+1,w}^{k}(w_{t+1}^{k})e^{W_{t+1}^{k}(w_{t+1}^{k})}\right]}{E_t\left[e^{W_{t+1}^{k}(w_{t+1}^{k})}\right]}.\end{aligned} \quad (14)$$

We proceed in two steps to show that (14) implies that $s_t^{k'} \geq s_t^k$ when k' is close to k. In the first one, we prove that the upper line of (14) has the same sign as $s_t^{k'} - s_t^k$, and in the second one that the bottom line of (14) is positive.

Let us start with the upper line of (14). Since u is concave and $k' > 0$, the term $k'(u'(c_t^{k'}) -$

$u'(c_t^k))$ has the same sign as $s_t^{k'} - s_t^k$. Moreover, we have:

$$\frac{E_t\left[W_{t+1,w}^{k'}(w_{t+1}^{k'})e^{W_{t+1}^{k'}(w_{t+1}^{k'})}\right]}{E_t\left[e^{W_{t+1}^{k'}(w_{t+1}^{k'})}\right]} - \frac{E_t\left[W_{t+1,w}^{k'}(w_{t+1}^{k})e^{W_{t+1}^{k'}(w_{t+1}^{k})}\right]}{E_t\left[e^{W_{t+1}^{k'}(w_{t+1}^{k})}\right]} =$$
$$\frac{E_t\left[\left(W_{t+1,w}^{k'}(w_t^{k'}) - W_{t+1,w}^{k'}(w_t^{k})\right)e^{W_{t+1}^{k'}(w_{t+1}^{k'})}\right]}{E_t\left[e^{W_{t+1}^{k'}(w_{t+1}^{k'})}\right]} + \left(J(w_{t+1}^{k'}) - J(w_{t+1}^{k})\right),$$

where J is the function: $w \to J(w) = E_t\left[W_{t+1,w}^{k'}(w_t^k)\frac{e^{W_{t+1}^{k'}(w)}}{E_t\left[e^{W_{t+1}^{k'}(w)}\right]}\right]$. Since $w \mapsto W_{t+1,w}^{k'}(w)$ is increasing (indeed $V_{t+1}^{k'} = -W_{t+1}^{k'}/k'$ is concave), $W_{t+1,w}^{k'}(w_t^{k'}) - W_{t+1,w}^{k'}(w_t^k)$ has the sign of $w_t^{k'} - w_t^k$ and therefore of $s_t^{k'} - s_t^k$. Moreover, $W_{t+1}^{k'}$ being continuously derivable, we can compute:

$$\frac{\partial J(w)}{\partial w} = cov_t^{\mathbb{Q}_{k'}}(W_{t+1,w}^{k'}(w_t^k), W_{t+1,w}^{k'}(w)), \tag{15}$$

where the probability $\mathbb{Q}_{k'}$ is defined by its Radon-Nikodym derivative: $\frac{d\mathbb{Q}_{k'}}{d\mathbb{P}} = \frac{e^{W_{t+1}^{k'}(w)}}{E_t\left[e^{W_{t+1}^{k'}(w)}\right]}$. The term (15) is positive when $w = w_t^k$ (the covariance is then a variance) and therefore is, by continuity, also positive for any w close to w_t^k. Since $k \mapsto s_t^k$ is continuous, we obtain that the term (15) is positive whenever k' is close to k. This implies that $J(w_{t+1}^{k'}) - J(w_{t+1}^k)$ has also the same sign as $s_t^{k'} - s_t^k$, as soon as k' is close to k.

We now focus on the bottom line of (14). The bottom line of (14) can be rewritten as $h(k, s_t^k) - h(k', s_t^k)$, where $h : (k, s_t^k) \mapsto \frac{ku'(c_t^k)}{\beta R_{t+1}} + \frac{E_t\left[W_{t+1,w}^{k}(w_{t+1}^{k})e^{W_{t+1}^{k}(w_{t+1}^{k})}\right]}{E_t\left[e^{W_{t+1}^{k}(w_{t+1}^{k})}\right]}$. When $k' > k$ remains close to k, the sign of $h(k, s_t^k) - h(k', s_t^k)$ is the opposite of the one of $\left.\frac{\partial h(k,s_t^k)}{\partial k}\right|_{s_t^k=cst}$. Since (11) implies that $\left.\frac{\partial W_{t+1,w}^k}{\partial k}\right|_{s_t^k} = -u'(c_{t+1}^k)$, we have:

$$\left.\frac{\partial E_t^{\mathbb{Q}_k}\left[W_{t+1,w}^k\right]}{\partial k}\right|_{s_t^k} = -E_t^{\mathbb{Q}_k}\left[u'(c_{t+1}^k)\right] + E_t^{\mathbb{Q}_k}\left[W_{t+1,w}^k W_{t+1,k}^k\right] - E_t^{\mathbb{Q}_k}\left[W_{t+1,w}^k\right]E_t^{\mathbb{Q}_k}\left[W_{t+1,k}^k\right],$$

where the probability $\mathbb{Q}_k$ is defined similarly to $\mathbb{Q}_{k'}$. Using (10) and (11) we get:

$$\left.\frac{\partial h(k, s_t^k)}{\partial k}\right|_{s_t^k} = -k\beta R_{t+1}\, cov_t^{\mathbb{Q}_k}(u'(c_{t+1}^k), W_{t+1,k}^k). \tag{16}$$

Observe now that, from equation (12), we can derive by iteration:

$$W^k_{t+1,k} = -\sum_{\tau=1}^{\infty}\beta^{\tau-1}E_{t+1}\left[\frac{u(c^k_{t+\tau})e^{\sum_{j=2}^{\tau}W^k_{t+j}}}{\prod_{j=2}^{\tau}E_{t+j-1}\left[e^{W^k_{t+j}}\right]}\right] = -\sum_{\tau=1}^{\infty}\beta^{\tau-1}E^{\widehat{\mathbb{Q}}_{t+\tau}}_{t+1}\left[u(c^k_{t+\tau})\right], \qquad (17)$$

where for any $\tau \geq 1$, the probability $\widehat{\mathbb{Q}}_{t+\tau}$ is defined by its Radon-Nikodym derivative: $\frac{d\widehat{\mathbb{Q}}_{t+\tau}}{d\mathbb{P}} = \frac{e^{\sum_{j=2}^{\tau}W^k_{t+j}}}{\prod_{j=2}^{\tau}E_{t+j-1}\left[e^{W^k_{t+j}}\right]}$. Using this notation, we finally obtain:

$$\left.\frac{\partial h(k, s^k_t)}{\partial k}\right|_{s^k_t} = k\beta R_{t+1}\sum_{\tau=1}^{\infty}\beta^{\tau-1}cov^{\widehat{\mathbb{Q}}_{t+\tau}}_t\left(u'(c^k_{t+1}), u(c^k_{t+\tau})\right). \qquad (18)$$

Lemma 1 implies that $(c^k_t)_{t\geq 0}$ and $(y_t)_{t\geq 0}$ are conditionally comonotonic. Proposition 2 implies then that $cov^{\widehat{\mathbb{Q}}_{t+\tau}}_t\left(u'(c^k_{t+1}), u(c^k_{t+\tau})\right) < 0$, or using (18), that $\left.\frac{\partial h(k,s^k_t)}{\partial k}\right|_{s^k_t} < 0$. We deduce that for $k' > k$ close to k, the the bottom line of (14) is positive.

In this case, we can therefore conclude that $s^{k'}_t \geq s^k_t$, whenever $k' \geq k$ is close to k.

Second case: Euler equations (10) and (13) hold with strict inequality for agent k. We have $s^k_t = \underline{s}_t$. The budget constraint implies that $s^{k'}_t \geq \underline{s}_t = s^k_t$.

Conclusion. We have shown that $k \mapsto s^k_t$ is locally non-decreasing for any $k \geq 0$, which implies that $k \mapsto s^k_t$ is globally non-decreasing on $\mathbb{R}^+$. This concludes the proof.

References

Ahn, C. M. (1989): "The Effect of Temporal Risk Aversion on Optimal Consumption, the Equity Premium, and the Equilibrium Interest Rate," *Journal of Finance*, 44(5), 1411–1420.

Araujo, A. (1991): "The Once But Not Twice Differentiability of the Policy Function," *Econometrica*, 59(5), 1383–1393.

Arrow, K. J. (1963): "Liquidity preference," Lecture VI in *Lecture Notes for Economics 285, The Economics of Uncertainty*, pp. 33–53, undated, Stanford University.

Bates, T. W., K. M. Kahle, and R. M. Stulz (2009): "Why do U.S. Firms Hold so Much More Cash Than They Used To?," *The Journal of Finance*, 64(5), 1985–2021.

Benveniste, L. M., and J. A. Scheinkman (1979): "On the Differentiability of the Value Function in Dynamic Models of Economics," *Econometrica*, 47(3), 727–732.

Bleichrodt, H., and L. Eeckhoudt (2005): "Saving under Rank-Dependent Utility," *Economic Theory*, 25(2), 505–511.

Bommier, A. (2013): "Life Cycle Preferences Revisited," *Journal of European Economic Association*, 11(6), 1290–1319.

Bommier, A., A. Chassagnon, and F. LeGrand (2012): "Comparative Risk Aversion: A Formal Approach with Applications to Saving Behaviors," *Journal of Economic Theory*, 147(4), 1614–1641.

Bommier, A., A. Kochov, and F. LeGrand (2017): "On Monotone Recursive Preferences," *Econometrica*, Forthcoming.

Campello, M., E. Giambona, J. R. Graham, and C. R. Harvey (2011): "Liquidity Management and Corporate Investment During a Financial Crisis," *Review of Financial Studies*, 24(6), 1944–1979.

Campello, M., J. R. Graham, and C. R. Harvey (2010): "The Real Effects of Financial Constraints: Evidence from a Financial Crisis," *Journal of Financial Economics*, 97(3), 470–487.

Carroll, C. D. (1997): "Buffer-Stock Saving and the Life Cycle/Permanent Income Hypothesis," *Quarterly Journal of Economics*, 112(1), 1–55.

Challe, E., and X. Ragot (2016): "Precautionary Saving over the Business Cycle," *Economic Journal*, 126(590), 135–164.

Chew, S. H., and L. G. Epstein (1991): "Recursive Utility Under Uncertainty," in *Equilibrium Theory in Infinite Dimensional Spaces*, ed. by A. M. Khan, and N. C. Yannelis, vol. 1 of *Studies in Economic Theory*, pp. 352–369. Springer Berlin Heidelberg.

Chiu, W. H. (2005): "Skewness Preference, Risk Aversion, and the Precedence Relations on Stochastic Changes," *Management Science*, 51(12), 1816–1828.

Courbage, C., and B. Rey (2007): "Precautionary Saving in the Presence of Other Risks," *Economic Theory*, 32(2), 414–424.

Denuit, M. M., and L. Eeckhoudt (2010): "A General Index of Absolute Risk Attitude," *Management Science*, 56(4), 712–715.

Drèze, J. H., and F. Modigliani (1972): "Consumption Decisions under Uncertainty," *Journal of Economic Theory*, 5(3), 308–335.

Duchin, R. (2010): "Cash Holdings and Corporate Diversification," *Journal of Finance*, 65(3), 955–992.

Duchin, R., O. Ozbas, and B. A. Sensoy (2010): "Costly External Finance, Corporate Investment, and the Subprime Mortgage Credit Crisis," *Journal of Financial Economics*, 97(3), 418–435.

Duncan, G. T. (1977): "A Matrix Measure of Multivariate Local Risk Aversion," *Econometrica*, 45(4), 895–903.

Ebert, S., and D. Wiesen (2011): "Testing for Prudence and Skewness Seeking," *Management Science*, 57(7), 1334–1349.

Eeckhoudt, L., and H. Schlesinger (2008): "Changes in Risk and the Demand for Saving," *Journal of Monetary Economics*, 55(7), 1329–1336.

Epstein, L. G., and S. E. Zin (1989): "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework," *Econometrica*, 57(4), 937–969.

Hansen, L. P., and T. J. Sargent (1995): "Discounted Linear Exponential Quadratic Gaussian Control," *IEEE Transactions on Automatic Control*, 40(5), 968–971.

Hansen, L. P., T. J. Sargent, G. Turmuhambetova, and N. Williams (2006): "Robust Control and Model Misspecification," *Journal of Economic Theory*, 128(1), 45–90.

Jouini, E., and C. Napp (2004): "Conditional Comonotonicity," *Decisions in Economics* *and Finance*, 27(2), 153–166.

Jouini, E., C. Napp, and D. C. Nocetti (2013): "On Multivariate Prudence," *Journal of Economic Theory*, 148(3), 1255–1267.

Karni, E. (1979): "On Multivariate Risk Aversion," *Econometrica*, 47(6), 1391–1401.

Kihlstrom, R. E., and L. J. Mirman (1974): "Risk Aversion with many Commodities," *Journal of Economic Theory*, 8(3), 361–388.

Kimball, M. (1990): "Precautionary Savings in the Small and in the Large," *Econometrica*, 58(1), 53–73.

Kimball, M., and P. Weil (2009): "Precautionary Saving and Consumption Smoothing Across Time and Possibilities," *Journal of Money, Credit, and Banking*, 41(2-3), 245–284.

Kreps, D. M., and E. L. Porteus (1978): "Temporal Resolution of Uncertainty and Dynamic Choice Theory," *Econometrica*, 46(1), 185–200.

Leland, H. E. (1968): "Saving and Uncertainty: The Precautionary Demand for Saving," *Quarterly Journal of Economics*, 82(3), 465–473.

Light, B. (2016): "Precautionary Saving in a Markovian Earnings Environment," Working paper, Stanford University.

Lins, K. V., H. Servaes, and P. Tufano (2010): "What Drives Corporate Liquidity? An International Survey of Cash Holdings and Lines of Credit," *Journal of Financial Economics*, 98(1), 160–176.

Menezes, C., C. Geiss, and J. Tressler (1980): "Increasing Downside Risk," *American Economic Review*, 70(5), 921–932.

Miller, B. L. (1976): "The Effect on Optimal Consumption of Increased Uncertainty in Labor Income in the Multiperiod Case," *Journal of Economic Theory*, 13(1), 154–167.

Mody, A., D. Sandri, and F. Ohnsorge (2012): "Precautionary Savings in the Great Recession," *IMF Economic Review*, 60(1), 114–138.

Nocetti, D. C. (2016): "Robust Comparative Statics of Risk Changes," *Management Science*, 62(5), 1381–1392.

Pratt, J. W. (1964): "Risk Aversion in the Small and in the Large," *Econometrica*, 32(1-2), 122–136.

Pye, G. (1973): "Lifetime Portfolio Selection in Continuous Time for a Multiplicative Class of Utility Functions," *American Economic Review*, 63(5), 1013–1016.

Richard, S. F. (1975): "Multivariate Risk Aversion, Utility Independence and Separable Utility Functions," *Management Science*, 22(1), 12–21.

Sandmo, A. (1970): "The Effect of Uncertainty on Saving Decisions," *Review of Economic Studies*, 37(3), 353–360.

Santos, M. S. (1991): "Smoothness of the Policy Function in Discrete Time Economic Models," *Econometrica*, 59(5), 1365–1382.

Stokey, N. L., R. E. J. Lucas, and E. C. Prescott (1989): *Recursive Methods in Economic Dynamics*. Harvard University Press.

Tallarini, T. D. J. (2000): "Risk-Sensitive Real Business Cycles," *Journal of Monetary Economics*, 45(3), 507–532.

Tauchen, G. (1986): "Finite State Markov-Chain Approximations to Univariate and Vector Autoregressions," *Economics Letters*, 20(2), 177–181.

van der Ploeg, F. (1993): "A Closed-Form Solution for a Model of Precautionary Saving," *Review of Economic Studies*, 60(2), 385–395.

Wang, C., N. Wang, and J. Yang (2016): "Optimal Consumption and Savings with Stochastic Income and Recursive Utility," *Journal of Economic Theory*, 165, 292–331.

Weil, P. (1990): "Non-Expected Utility in Macroeconomics," *Quarterly Journal of Economics*, 105(1), 29–42.

——— (1993): "Precautionary Savings and the Permanent Income Hypothesis," *Review of Economic Studies*, 60(2), 367–383.

Yaari, M. E. (1969): "Some Remarks on Measures of Risk Aversion and their Uses," *Journal of Economic Theory*, 1(3), 315–329.