Technical appendix for "Incomplete markets, liquidation risk, and the term structure of interest rates"

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Introduction

This technical appendix completes the paper "Incomplete markets, liquidation risk, and the term structure of interest rates" and is organised in three sections.

- Section 1 (Page 2) offers a step-by-step derivation of all the paper's proofs.
- Section 2 (Page 35) constructs, and then studies quantitatively, a relaxed model where a number of assumptions of the baseline theoretical model are removed.
- Section 3 (Page 42) provides additional theoretical results. In particular, we analyse
  - the robustness of our results with respect to an alternative taxation scheme,
  - the impact of bond supplies on welfare,
  - the implications of our baseline model for time-variations in risk premia and the rejection of the Expectation Hypothesis,
  - a model variant with log preferences and growth-stationary productivity,
  - a model variant with an alternative specification for the borrowing constraint.

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1 Detailed proofs of paper's results

1.1 Proof of Proposition 1

1. Yield of infinite-maturity bonds. We know that in the complete market economy, the $k-$period interest rate is defined as follows:

$$r_{t,k}^{CM} = -\ln(\beta) - \frac{1}{k}\left(\ln\left(E_t\left[z_{t+k}^{-1}\right]\right) - \ln(z_t^{-1})\right). \tag{1}$$

Since z_t is bounded, we have $\lim_{k\to\infty} r_{t,k} = -\ln\beta \equiv r_\infty^{CM}$ (independent of the aggregate state).

2. Monotonicity of conditional yield curves. Let us denote $T^k = (T_{ij}^k)_{i,j=l,h}$, where T_{ij}^k is the probability of being in aggregate state j k periods ahead when the current state is i, i.e. the ij's element of T^k (T to the power of k), where

$$T = \begin{bmatrix} \pi^h & 1-\pi^h \\ 1-\pi^l & \pi^l \end{bmatrix}. \tag{2}$$

By the definition of T^k, we have $E_t\left[z_{t+k}^{-1}\right] = T_{hh}^k \frac{1}{z^h} + \left(1-T_{hh}^k\right)\frac{1}{z^l}$ if the current state is h, while $E_t\left[z_{t+k}^{-1}\right] = T_{ll}^k \frac{1}{z^l} + \left(1-T_{ll}^k\right)\frac{1}{z^h}$ if the current state is l. We infer from (1) the expressions for the yield differences $r_\infty^{CM} - r_k^s$, $s=l,h$:

$$r_\infty^{CM} - r_k^h = \frac{1}{k}\ln\left[T_{hh}^k + \left(1-T_{hh}^k\right)\frac{z^h}{z^l}\right], \tag{3}$$

$$r_\infty^{CM} - r_k^l = \frac{1}{k}\ln\left[T_{ll}^k + \left(1-T_{ll}^k\right)\frac{z^l}{z^h}\right]. \tag{4}$$

In order to analyse the shape of conditional yield curve, we express the sequences $\{T_{hh}^k\}_{k=1}^\infty$ and $\{T_{ll}^k\}_{k=1}^\infty$ as follows:

$$(T_{hh}^1, T_{ll}^1) = \left(\pi^h, \pi^l\right) \text{ and } \left(T_{hh}^{k+1}, T_{hh}^{k+1}\right) = \left(T_{hh}^k\left(\pi^h+\pi^l-1\right) + 1-\pi^l, T_{ll}^k\left(\pi^h+\pi^l-1\right)+1-\pi^h\right). \tag{5}$$

Assumption A, which imposes that $\pi^h + \pi^l - 1 > 0$, implies that the sequences $\{T_{hh}^k\}_{k=1}^\infty$ and $\{T_{ll}^k\}_{k=1}^\infty$ are nondecreasing and belong to the open interval $]0;1[$. Because $z^h > z^l$, this implies that $T_{hh}^k + \left(1-T_{hh}^k\right)\frac{z^h}{z^l}$ is nonincreasing in k. We deduce that $k \mapsto \frac{1}{k}\ln\left[T_{hh}^k + \left(1-T_{hh}^k\right)\frac{z^h}{z^l}\right]$ is the product of two nonincreasing and positive functions. Therefore, $r_\infty^{CM} - r_k^h$ is strictly decreasing in k and converges to 0 for large k. Hence, r_k^h lies below r_∞^{CM} and is strictly increasing in k. A symmetric argument applies to the yield curve in state l.

3. Monotonicity of the average yield curve. We define the average yield $\bar{r}_k$ as the unconditional mean of conditional yields: $\bar{r}_k = (1-\eta^l)r_k^h + \eta^l r_k^l$, where $\eta^l = \left(1-\pi^h\right)/\left(2-\pi^h-\pi^l\right)$ is the unconditional probability of being in state l. From Equations (3) and (4), the average yield difference $\psi(k) \equiv \bar{r}_k - r_\infty^{CM}$ is given by:

$$\psi(k) = -\frac{1-\eta^l}{k}\ln\left(T_{hh}^k + \left(1 - T_{hh}^k\right)\frac{z^h}{z^l}\right) - \frac{\eta^l}{k}\ln\left(T_{ll}^k + \left(1 - T_{ll}^k\right)\frac{z^l}{z^h}\right). \tag{6}$$

Diagonalising T, we can prove by induction that we can express $\left(T_{ll}^k, T_{hh}^k\right)$ as follows:

$$T_{hh}^k = \frac{(1-\pi^h)(\pi^h+\pi^l-1)^k + 1 - \pi^l}{2-\pi^h-\pi^l} \text{ and } T_{ll}^k = \frac{(1-\pi^l)(\pi^h+\pi^l-1)^k + 1 - \pi^h}{2-\pi^h-\pi^l}. \tag{7}$$

Take T_{hh}^k first. We have, for $k=1$: $T_{hh}^1 = \frac{(1-\pi^h)(\pi^h+\pi^l-1)+1-\pi^l}{2-\pi^h-\pi^l} = \pi^h$. Then, using (5) we have $T_{hh}^{k+1} = T_{hh}^k\left(\pi^h+\pi^l-1\right) + 1 - \pi^l = \frac{(1-\pi^h)(\pi^h+\pi^l-1)^{k+1}+(1-\pi^l)(\pi^h+\pi^l-1)+(1-\pi^l)(2-\pi^h-\pi^l)}{2-\pi^h-\pi^l} = \frac{(1-\pi^h)(\pi^h+\pi^l-1)^{k+1}+1-\pi^l}{2-\pi^h-\pi^l}$. Hence, this is true for any $k>0$. A similar argument applies for T_{ll}^k.

Substituting (7) into (6), we can write $\psi(k) = -\phi(k)/k$, where:

$$\begin{aligned}
\phi(k) &= \frac{1-\pi^l}{2-\pi^h-\pi^l}\ln\left(\frac{(1-\pi^h)(\pi^h+\pi^l-1)^k(1-\frac{z^h}{z^l}) + (1-\pi^l) + (1-\pi^h)\frac{z^h}{z^l}}{2-\pi^h-\pi^l}\right) \\
&+ \frac{1-\pi^h}{2-\pi^h-\pi^l}\ln\left(\frac{(1-\pi^l)(\pi^h+\pi^l-1)^k(1-\frac{z^l}{z^h}) + (1-\pi^h) + (1-\pi^l)\frac{z^l}{z^h}}{2-\pi^h-\pi^l}\right) \\
&= (1-\eta^l)\ln\left(1 + \eta^l\left((\pi^h+\pi^l-1)^k - 1\right)(1-\frac{z^h}{z^l})\right) \\
&+ \eta^l\ln\left(1 + (1-\eta^l)\left((\pi^h+\pi^l-1)^k - 1\right)(1-\frac{z^l}{z^h})\right).
\end{aligned}$$

It is straightforward to note that:

$$k^2\psi'(k) = \phi(k) - k\phi'(k) \text{ and } \frac{\partial\left(k^2\psi'(k)\right)}{\partial k} = -\phi''(k).$$

Since $\phi(0)=0$, we infer that $\lim_{k\to 0} k^2\psi'(k) = 0$. Moreover, $\lim_{k\to\infty}\psi(k) = 0$. To study the sign and the variations of ψ, we take the derivative of ϕ twice with respect to k:

$$\begin{aligned}
\phi'(k) &= \eta^l(1-\eta^l)\ln(\pi^h+\pi^l-1) \\
&\times\left(\frac{(\pi^h+\pi^l-1)^k\left(1-\frac{z^h}{z^l}\right)}{1+\eta^l\left((\pi^h+\pi^l-1)^k-1\right)(1-\frac{z^h}{z^l})} + \frac{(\pi^h+\pi^l-1)^k\left(1-\frac{z^l}{z^h}\right)}{1+(1-\eta^l)\left((\pi^h+\pi^l-1)^k-1\right)(1-\frac{z^l}{z^h})}\right),
\end{aligned}$$

and:

$$
\begin{aligned}
\phi''(k) &= \eta^l(1-\eta^l)\left(\ln(\pi^h+\pi^l-1)\right)^2(\pi^h+\pi^l-1)^k \\
&\quad\times\left(\frac{\left(1-\eta^l\left(1-\frac{z^h}{z^l}\right)\right)\left(1-\frac{z^h}{z^l}\right)}{\left(1+\eta^l\left((\pi^h+\pi^l-1)^k-1\right)(1-\frac{z^h}{z^l})\right)^2}+\frac{\left(1-(1-\eta^l)\left(1-\frac{z^l}{z^h}\right)\right)\left(1-\frac{z^l}{z^h}\right)}{\left(1+(1-\eta^l)\left((\pi^h+\pi^l-1)^k-1\right)(1-\frac{z^l}{z^h})\right)^2}\right) \\
&= \eta^l(1-\eta^l)\left(\ln(\pi^h+\pi^l-1)\right)^2(\pi^h+\pi^l-1)^k(z^h-z^l)\left(\frac{\eta^l}{z^l}+\frac{1-\eta^l}{z^h}\right)z^hz^l \\
&\quad\times\left(\frac{-1}{(z^l-\eta^l\left((\pi^h+\pi^l-1)^k-1\right)(z^h-z^l))^2}+\frac{1}{(z^h+(1-\eta^l)\left((\pi^h+\pi^l-1)^k-1\right)(z^h-z^l))^2}\right) \\
&= \eta^l(1-\eta^l)\left(\ln(\pi^h+\pi^l-1)\right)^2(\pi^h+\pi^l-1)^k(z^h-z^l)\left(\frac{\eta^l}{z^l}+\frac{1-\eta^l}{z^h}\right)z^hz^l \\
&\quad\times\frac{\left(z^l-z^h-\left((\pi^h+\pi^l-1)^k-1\right)(z^h-z^l)\right)\left(z^h+z^l+(1-2\eta^l)\left((\pi^h+\pi^l-1)^k-1\right)(z^h-z^l)\right)}{(z^l+\eta^l\left((\pi^h+\pi^l-1)^k-1\right)(z^l-z^h))^2\,(z^h+(1-\eta^l)\left((\pi^h+\pi^l-1)^k-1\right)(z^h-z^l))^2} \\
&= \eta^l(1-\eta^l)\left(\ln(\pi^h+\pi^l-1)\right)^2(\pi^h+\pi^l-1)^{2k}(z^h-z^l)^2\left(\frac{\eta^l}{z^l}+\frac{1-\eta^l}{z^h}\right)z^hz^l \\
&\quad\times\frac{-\left(z^h+z^l+(1-2\eta^l)\left((\pi^h+\pi^l-1)^k-1\right)(z^h-z^l)\right)}{(z^l+\eta^l\left((\pi^h+\pi^l-1)^k-1\right)(z^l-z^h))^2\,(z^h+(1-\eta^l)\left((\pi^h+\pi^l-1)^k-1\right)(z^h-z^l))^2} \\
&= -\eta^l(1-\eta^l)\left(\ln(\pi^h+\pi^l-1)\right)^2(\pi^h+\pi^l-1)^{2k}(z^h-z^l)^2\left(\frac{\eta^l}{z^l}+\frac{1-\eta^l}{z^h}\right)z^hz^l \\
&\quad\times\frac{2(\eta^lz^h+(1-\eta^l)z^l)\left(1-(\pi^h+\pi^l-1)^k\right)+(\pi^h+\pi^l-1)^k(z^h+z^l)}{(z^l+\eta^l\left((\pi^h+\pi^l-1)^k-1\right)(z^l-z^h))^2\,(z^h+(1-\eta^l)\left((\pi^h+\pi^l-1)^k-1\right)(z^h-z^l))^2}.
\end{aligned}
$$

Using Assumption A, which implies that, for all $k\geq 0$, $0\leq(\pi^h+\pi^l-1)^k\leq 1$, we infer that $\frac{\partial\left(k^2\psi'(k)\right)}{\partial k}=-\phi''(k)\geq 0$. Since $\lim_{k\to 0}k^2\psi'(k)=0$, we have $\psi'(k)\geq 0$ for all k. This implies that ψ is increasing and negative on $\mathbb{R}^+$ (because $\psi(\infty)=0$). This implies that $\bar{r}_k$ lies below r_∞^{CM} and is strictly increasing in k.

1.2 Proof of Proposition 2

1. Pricing kernel From Equations **(12)**–**(13)** in the paper, we find that

$$
p_{t,k}=\beta E_t\left[\left(\alpha_{t+1}\frac{z_t}{z_{t+1}}+(1-\alpha_{t+1})\,z_tu'(\delta)\right)p_{t+1,k-1}\right],\quad k=1,\ldots,n,
$$

which provides the pricing kernel factorisation in m_{t+1}^{ZV} and I_{t+1}^{ZV}. From the literature on asset pricing with finite state-space (e.g., Mehra and Prescott (1985)), we conjecture (and verify) the existence of an equilibrium in which bond prices at any date t only depend on the current aggregate state s_t (and not on the whole history s^t), so that i.e., $p_{t,k}(s^t)=p_{t,k}$. With two aggregate states, bond prices are generated by the following recursions:

$$
\frac{p_k^s}{z^s}=\beta\pi^s\left(\alpha^s+(1-\alpha^s)\,z^su'(\delta)\right)\frac{p_{k-1}^s}{z^s}+\beta\left(1-\pi^s\right)\left(\alpha^{\bar{s}}+\left(1-\alpha^{\bar{s}}\right)z^{\bar{s}}u'(\delta)\right)\frac{p_{k-1}^{\bar{s}}}{z^{\bar{s}}},\ s=l,h, \tag{8}
$$

for $k=1,...n$ and where $\bar{s}$ is the state opposite to s. At these prices unemployed agents face a binding borrowing constraint in state s if and only if:

$$p_k^s u'(\delta) > \beta \left(\pi^s \left(\frac{1-\rho^s}{z^s} + \rho^s u'(\delta) \right) p_{k-1}^s + (1-\pi^s) \left(\frac{1-\rho^{\bar{s}}}{z^{\bar{s}}} + \rho^{\bar{s}} u'(\delta) \right) p_{k-1}^{\bar{s}} \right), \; s = l, h.$$

Assumption C is a sufficient condition for these two inequalities to be satisfied when bond prices satisfy (8), so that unemployed agents do not participate in bond markets (i.e., they would like to *issue* bonds, but face a binding borrowing constraint, in both aggregate state).

2. Convergence towards a common limit. We first prove the following technical lemma.

Lemma 1 *Let $(u_n)_{n\geq 0}$, $(v_n)_{n\geq 0}$ be two real sequences such that $[\begin{array}{cc} u_n & v_n \end{array}]^\top = M[\begin{array}{cc} u_{n-1} & v_{n-1} \end{array}]^\top$, where M is a 2×2 real diagonalisable matrix whose eigenvalues λ_{max} and λ_{min} are positive. Then, $\left(-n^{-1}\ln(u_n)\right)_{n\geq 0}$ and $\left(-n^{-1}\ln(v_n)\right)_{n\geq 0}$ converge towards the common limit $\ln(\lambda_{max})$.*

Proof: Diagonalising M, we may rewrite the recursion in the Lemma as

$$\left[\begin{array}{c} u_n \\ v_n \end{array}\right] = Q \left[\begin{array}{cc} \lambda_{max}^n & 0 \\ 0 & \lambda_{min}^n \end{array}\right] Q^{-1} \left[\begin{array}{c} u_0 \\ v_0 \end{array}\right],$$

where $(\lambda_{max}, \lambda_{min})$, $\lambda_{max} \geq \lambda_{min} \geq 0$ and λ_{max}, are then eigenvalues of M and Q the matrix of eigenvectors. Making the matrix products, it is straightforward that $\frac{u_n}{\lambda_{max}^n}$ and $\frac{v_n}{\lambda_{max}^n}$ are affine functions of $\left(\frac{\lambda_{min}}{\lambda_{max}}\right)^n$, which is positive and either is equal to 1 or converges towards 0 as $n \to \infty$. Thus, $\frac{u_n}{\lambda_{max}^n}$ and $\frac{v_n}{\lambda_{max}^n}$ converge towards finite limits and are bounded for all n. We infer that $-\frac{1}{n}\ln(\frac{u_n}{\lambda_{max}^n})$ and $-\frac{1}{n}\ln(\frac{v_n}{\lambda_{max}^n})$ converge toward 0, and that $\lim_{n\to\infty} -\frac{1}{n}\ln(u_n) = \lim_{n\to\infty} -\frac{1}{n}\ln(v_n) = \ln(\lambda_{max})$ □

We may rewrite the bond price recursion in (8) in matrix form as follows:

$$\left[\begin{array}{cc} p_k^h/z^h & p_k^l/z^l \end{array}\right]^\top = M^{ZV}[\begin{array}{cc} p_{k-1}^h/z^h & p_{k-1}^l/z^l \end{array}]^\top \text{ for } k \geq 1, \text{ with } p_0^s = 1, \tag{9}$$

$$M^{ZV} = \beta \left(\begin{array}{cc} \pi^h \kappa^h & \left(1-\pi^h\right)\kappa^l \\ \left(1-\pi^l\right)\kappa^h & \pi^l \kappa^l \end{array} \right) \text{ and } \kappa^s \equiv \alpha^s + (1-\alpha^s) z^s u'(\delta). \tag{10}$$

Then, Lemma 1 implies that $\lim_{k\to\infty} r_k^s = r_\infty^{ZV}$, $s = h, l$, where

$$r_\infty^{ZV} = -\ln(\beta) - \ln(\nu_1), \text{ and} \tag{11}$$

$$\nu_1 = \frac{1}{2}\left(\kappa^h \pi^h + \kappa^l \pi^l + \left((\kappa^h \pi^h + \kappa^l \pi^l)^2 - 4\kappa^h \kappa^l (\pi^h + \pi^l - 1)\right)^{\frac{1}{2}}\right). \tag{12}$$

The largest eigenvalue of M^{ZV} is denoted ν_1, while the other one is

$$\nu_2 = \frac{1}{2}\left(\kappa^h \pi^h + \kappa^l \pi^l - \left((\kappa^h \pi^h + \kappa^l \pi^l)^2 - 4\kappa^h \kappa^l (\pi^h + \pi^l - 1)\right)^{\frac{1}{2}}\right).$$

It is straightforward that $\nu_1 \geq \nu_2 \geq 0$.

3. Monotonicity of conditional yield curves. As stated in the proposition, a sufficient condition for the monotonicity of conditional yield curves in the incomplete-market, zero-volume case is that α^l and α^h be sufficiently close to each other. The necessary and sufficient condition is

$$\left(\pi^l + (\pi^h - 1)\frac{z^h}{z^l}\right)\left(\alpha^l + (1-\alpha^l)z^l u'(\delta)\right) \leq \left(\pi^h + (\pi^l - 1)\frac{z^l}{z^h}\right)\left(\alpha^h + (1-\alpha^h)z^h u'(\delta)\right), \tag{13}$$

and is indeed satisfied when α^l and α^h are close to each other, including when $\alpha^h = \alpha^l$. Under (13), the eigenvalues (ν_1, ν_2) of M^{ZV} in (12) satisfy

$$0 \leq \nu_2 \leq \nu_1 \leq \pi^h\kappa^h + (1-\pi^h)\left(z^h/z^l\right)\kappa^l, \tag{14}$$

$$\nu_2 \leq \pi^h\kappa^h \leq \nu_1. \tag{15}$$

Let us prove these inequalities:

- We already know that $0 \leq \nu_2 \leq \nu_1$. Moreover, using the definition of ν_1 in (12), the inequality $\nu_1 \leq \pi^h\kappa^h + (1-\pi^h)\frac{z^h}{z^l}\kappa^l$ is equivalent to:

$$\left((\kappa^h\pi^h + \kappa^l\pi^l)^2 - 4\kappa^h\kappa^l(\pi^h + \pi^l - 1)\right)^{\frac{1}{2}}$$

  which is itself equivalent to inequality (13). This concludes the proof of inequality (14).

- The inequality $\nu_1 \geq \pi^h\kappa^h$ becomes after substitution of the expression (12) of ν_1:

$$(\kappa^h\pi^h + \kappa^l\pi^l)^2 - 4\kappa^h\kappa^l(\pi^h + \pi^l - 1) \geq (\kappa^h\pi^h - \kappa^l\pi^l)^2$$
$$\kappa^h\pi^h\kappa^l\pi^l \geq \kappa^h\kappa^l(\pi^h + \pi^l - 1)$$
$$(1-\pi^h)(1-\pi^l) \geq 0,$$

  which always holds. A similar argument applies for $\nu_2 \leq \pi^h\kappa^h$ and establishes inequality (15).

From (9), bond prices are given by $\begin{bmatrix} p_k^h/z^h & p_k^l/z^l \end{bmatrix}^\top = M^{ZV,k}\begin{bmatrix} 1/z^h & 1/z^l \end{bmatrix}^\top$, where $M^{ZV,k} \equiv \left(M^{ZV}\right)^k$ (M^{ZV} to the k) is diagonalisable and can be written as

$$M^{ZV,k} = \beta^k P\begin{bmatrix} \nu_1^k & 0 \\ 0 & \nu_2^k \end{bmatrix}P^{-1}, \text{ with } P = \begin{bmatrix} (1-\pi^h)\kappa^l & (1-\pi^h)\kappa^l \\ \nu_1 - \pi^h\kappa^h & \nu_2 - \pi^h\kappa^h \end{bmatrix},$$

$$P^{-1} = \frac{1}{(1-\pi^h)\kappa^l(\nu_2-\nu_1)}\begin{bmatrix} \nu_2 - \pi^h\kappa^h & -(1-\pi^h)\kappa^l \\ -\nu_1 + \pi^h\kappa^h & (1-\pi^h)\kappa^l \end{bmatrix}.$$

We use the same strategy as when proving the monotonicity of the average yield curve in the complete-market case (See the function ψ defined in (6)). From the definition of $M^{ZV,k}$ we have:

$$M^{ZV,k}\begin{bmatrix}\frac{1}{z^h}\\ \frac{1}{z^l}\end{bmatrix} = \frac{-\beta^k}{(1-\pi^h)\kappa^l(\nu_1-\nu_2)}\begin{bmatrix}(1-\pi^h)\kappa^l & (1-\pi^h)\kappa^l\\ \nu_1-\pi^h\kappa^h & \nu_2-\pi^h\kappa^h\end{bmatrix}\begin{bmatrix}\nu_1^k & 0\\ 0 & \nu_2^k\end{bmatrix}\begin{bmatrix}\nu_2-\pi^h\kappa^h & -(1-\pi^h)\kappa^l\\ -\nu_1+\pi^h\kappa^h & (1-\pi^h)\kappa^l\end{bmatrix}\begin{bmatrix}\frac{1}{z^h}\\ \frac{1}{z^l}\end{bmatrix}$$

$$= \frac{-\beta^k}{(1-\pi^h)\kappa^l(\nu_1-\nu_2)}\begin{bmatrix}(1-\pi^h)\kappa^l\nu_1^k & (1-\pi^h)\kappa^l\nu_2^k\\ \left(\nu_1-\pi^h\kappa^h\right)\nu_1^k & \left(\nu_2-\pi^h\kappa^h\right)\nu_2^k\end{bmatrix}\begin{bmatrix}\left(\nu_2-\pi^h\kappa^h\right)\frac{1}{z^h}-(1-\pi^h)\frac{\kappa^l}{z^l}\\ -\left(\nu_1-\pi^h\kappa^h\right)\frac{1}{z^h}+(1-\pi^h)\frac{\kappa^l}{z^l}\end{bmatrix}$$

$$= \frac{\beta^k\nu_1^k}{(1-\pi^h)\kappa^l(\nu_1-\nu_2)}\begin{bmatrix}(1-\pi^h)\kappa^l\left(-\left(\left(\nu_2-\pi^h\kappa^h\right)\frac{1}{z^h}-(1-\pi^h)\frac{\kappa^l}{z^l}\right)+\frac{\nu_2^k}{\nu_1^k}\left(\left(\nu_1-\pi^h\kappa^h\right)\frac{1}{z^h}-(1-\pi^h)\frac{\kappa^l}{z^l}\right)\right)\\ -\left(\nu_1-\pi^h\kappa^h\right)\left(\left(\nu_2-\pi^h\kappa^h\right)\frac{1}{z^h}-(1-\pi^h)\frac{\kappa^l}{z^l}\right)+\frac{\nu_2^k}{\nu_1^k}\left(\nu_2-\pi^h\kappa^h\right)\left(\left(\nu_1-\pi^h\kappa^h\right)\frac{1}{z^h}-(1-\pi^h)\frac{\kappa^l}{z^l}\right)\end{bmatrix}.$$

Using the definition (11) of r_∞^{ZV}, we infer that $r_k^h - r_\infty^{ZV} = \tilde{\psi}(k) = \tilde{\phi}(k)/k$, where

$$\tilde{\phi}(k) = \ln\left[\nu_1-\nu_2\right] - \ln\left[\frac{\nu_2^k}{\nu_1^k}\left(\left(\nu_1-\pi^h\kappa^h\right)-(1-\pi^h)\frac{z^h}{z^l}\kappa^l\right)-\left(\left(\nu_2-\pi^h\kappa^h\right)-(1-\pi^h)\frac{z^h}{z^l}\kappa^l\right)\right].$$

We have $k^2\tilde{\psi}'(k) = \tilde{\phi}(k) - k\tilde{\phi}'(k)$ and $\frac{\partial\left(k^2\tilde{\psi}'(k)\right)}{\partial k} = -\tilde{\phi}''(k)$. Since $\tilde{\phi}(0) = 0$, we deduce that $\lim_{k\to 0} k^2\tilde{\psi}'(k) = 0$. Moreover, $\lim_{k\to\infty}\tilde{\psi}(k) = 0$ since $0 \le \frac{\nu_2}{\nu_1} \le 1$. To study the sign and the variations of $\tilde{\psi}$, compute the second derivative of $\tilde{\phi}$ with respect to k:

$$\tilde{\phi}'(k) = \frac{\ln\left[\nu_2/\nu_1\right]\frac{\nu_2^k}{\nu_1^k}}{\frac{\left(\nu_2-\pi^h\kappa^h\right)-(1-\pi^h)\frac{z^h}{z^l}\kappa^l}{\left(\nu_1-\pi^h\kappa^h\right)-(1-\pi^h)\frac{z^h}{z^l}\kappa^l}-\frac{\nu_2^k}{\nu_1^k}},$$

and:

$$\tilde{\phi}''(k) = \frac{\left(\ln\left[\nu_2/\nu_1\right]\right)^2\frac{\nu_2^k}{\nu_1^k}}{\left(\frac{\left(\nu_2-\pi^h\kappa^h\right)-(1-\pi^h)\frac{z^h}{z^l}\kappa^l}{\left(\nu_1-\pi^h\kappa^h\right)-(1-\pi^h)\frac{z^h}{z^l}\kappa^l}-\frac{\nu_2^k}{\nu_1^k}\right)^2}\left(\frac{\left(\nu_2-\pi^h\kappa^h\right)-(1-\pi^h)\frac{z^h}{z^l}\kappa^l}{\left(\nu_1-\pi^h\kappa^h\right)-(1-\pi^h)\frac{z^h}{z^l}\kappa^l}\right).$$

Condition (13) implies that $\frac{\partial\left(k^2\tilde{\psi}'(k)\right)}{\partial k} = -\tilde{\phi}''(k) \ge 0$ for all $k \ge 0$. Together with $\lim_{k\to 0} k^2\tilde{\psi}'(k) = 0$, we have $\tilde{\psi}'(k) \ge 0$. Since $\tilde{\psi}(\infty) = 0$, we infer that $r_k^h - r_\infty^{ZV} = \tilde{\psi}(k)$ is positive and increasing. This implies that the yield curve is increasing in state h and converges from below to r_∞^{ZV}. A symmetric proof can be done to show that the yield curve is decreasing and converges to r_∞^{ZV} from above.

1.3 A preliminary result for the proof of Proposition 3

In the context of the economy with incomplete markets and zero net bond supply (Section **4.2** of the paper), our results pertaining the the impact of idiosyncratic risk on the slope of the yield curve are derived using second-order developments of the short and the long yield under small aggregate shocks. More specifically, we consider the following mean-preserving spread in aggregate risk:

$$z^h = z(1+2(1-\eta^h)\varepsilon) \quad \text{and} \quad z^l = z(1-2\eta^h\varepsilon), \tag{16}$$

where $\eta^h = \frac{1-\pi^l}{2-\pi^h-\pi^l}$ is the unconditional probability that the aggregate state is h. The unconditional mean is z, while the unconditional variance is $4\eta^h(1-\eta^h)z^2\varepsilon^2$. Similarly, for the idiosyncratic risk, we assume that

$$\alpha^h = \alpha(1 + 2(1-q)a) \quad \text{and} \quad \alpha^l = \alpha(1 - 2\eta^h a). \tag{17}$$

Both spreads a and ε are of the same order of magnitude, and we assume that $0 < \varepsilon, a \ll 1$.

We now prove the following lemma:

Lemma 2 (Second-order developments) *The short term interest rate $\overline{r}_1^{ZV}$ and the long term interest r_∞^{ZV} in the zero volume economy can be expressed at the second-order in the shocks a and ε as follows:*

$$\begin{aligned}\overline{r}_1^{ZV} &= -\ln\left(\beta\alpha + \beta(1-\alpha)zu'(\delta)\right) \\ &\quad + 2\frac{\alpha^2(1-\pi^h)(1-\pi^l)(\pi^h+\pi^l-1)^2(zu'(\delta)-1)^2}{(2-\pi^h-\pi^l)^2(\alpha+(1-\alpha)zu'(\delta))^2}a^2 \\ &\quad + \frac{4\alpha(1-\pi^h)(1-\pi^l)}{(2-\pi^h-\pi^l)^2(\alpha+(1-\alpha)zu'(\delta))^2}\left(zu'(\delta) - \alpha(2-\pi^h-\pi^l)(\pi^h+\pi^l)(zu'(\delta)-1)\right)a\varepsilon \\ &\quad + \frac{2(1-\pi^h)(1-\pi^l)}{(2-\pi^h-\pi^l)(\alpha+(1-\alpha)zu'(\delta))}\left(-2\alpha + \frac{\left((2-\pi^h-\pi^l)\alpha + (1-\alpha)zu'(\delta)\right)^2}{(2-\pi^h-\pi^l)(\alpha+(1-\alpha)zu'(\delta))}\right)\varepsilon^2,\end{aligned} \tag{18}$$

$$\begin{aligned}r_\infty^{ZV} &= -\ln\left(\beta(\alpha + (1-\alpha)zu'(\delta))\right) \\ &\quad - 4\frac{(\pi^h+\pi^l-1)(1-\pi^h)(1-\pi^l)}{(2-\pi^h+\pi^l)^3}\frac{\alpha^2(zu'(\delta)-1)^2a^2 + (1-\alpha)^2u'(\delta)^2z^2\varepsilon^2}{(\alpha+(1-\alpha)zu'(\delta))^2} \\ &\quad + 4\alpha\frac{(1-\pi^h)(1-\pi^l)zu'(\delta)}{(2-\pi^h+\pi^l)^2(\alpha+(1-\alpha)zu'(\delta))^2}\left(1 + \frac{\pi^h+\pi^l}{2-\pi^h-\pi^l}(1-\alpha)(zu'(\delta)-1)\right)a\varepsilon.\end{aligned} \tag{19}$$

The corresponding expression for the unconditional slope $\Delta^{ZV} = r_\infty^{ZV} - \overline{r}_1^{ZV}$ is:

$$\begin{aligned}\Delta^{ZV} &= \frac{2(1-\pi^h)(1-\pi^l)(\pi^h+\pi^l)}{(2-\pi^h+\pi^l)^3(\alpha+(1-\alpha)zu'(\delta))^2}\Big(\left((2-\pi^h-\pi^l)^2\alpha^2 - (1-\alpha)^2u'(\delta)^2z^2\right)\varepsilon^2 \\ &\quad - (\pi^h+\pi^l-1)(3-\pi^h+\pi^l)\alpha^2(zu'(\delta)-1)^2a^2 \\ &\quad + 2\alpha\left(\alpha(2-\pi^h-\pi^l)^2 + zu'(\delta)(1-\alpha)\right)(zu'(\delta)-1)a\varepsilon\Big).\end{aligned} \tag{20}$$

The proof of the lemma is done in three steps: (i) approximation to short yield, (ii) approximation to the long yield and (iii) implied slope of the yield curve. All expressions are second-order developments in a and ε. For the sake of conciseness, we skip the notations $O(\cdot)$ or $o(\cdot)$.

1.3.1 The short rate

In state $s = h, l$, the short rate is $r_1^{ZV,s} = -\ln(p_1^{ZV,s})$. The constant $C_1^{ZV,s}$ is defined such that $p_1^{ZV,s} = C_1^{ZV,s}\, z^s$. We proceed in three steps: we compute (i) the second-order developments of the constants $C_1^{ZV,s}$, (ii) the bond prices $p_1^{ZV,s}$, and (iii) the bond yields $r_1^{ZV,s}$.

Second-order development of $C_1^{ZV,s}$. We focus on $C_1^{ZV,h}$, but a symmetric development applies to $C_1^{ZV,l}$. From Equation **(20)** in the paper, we know that:[1]

$$C_1^{ZV,h} = \beta\left(\pi^h \frac{\alpha^h}{z^h} + (1-\pi^h)\frac{\alpha^l}{z^l} + \left(\pi^h(1-\alpha^h) + (1-\pi^h)(1-\alpha^l)\right) u'(\delta)\right).$$

Substituting the expressions for z^h, z^l in (16) and for α^h, α^l in (17) of , we obtain:

$$\begin{aligned}
\frac{C_1^{ZV,h}}{\beta} &= \frac{\alpha\pi^h}{z}\frac{1+2(1-\eta^h)a}{1+2(1-\eta^h)\varepsilon} + \frac{\alpha(1-\pi^h)}{z}\frac{1-2\eta^h a}{1-2\eta^h\varepsilon} + \left(1-\alpha-2\alpha(\pi^h(1-\eta^h)+(1-\pi^h)\eta^h)a\right)u'(\delta)\\
&= \frac{\alpha}{z}\left(\pi^h(1+2(1-\eta^h)a)(1-2(1-\eta^h)\varepsilon+4(1-\eta^h)^2\varepsilon^2) + (1-\pi^h)(1-2\eta^h a)(1+2\eta^h\varepsilon+4\eta^{h,2}\varepsilon^2)\right)\\
&+ (1-\alpha)u'(\delta) - 2\alpha((1-\eta^h)\pi^h - \eta^h(1-\pi^h))u'(\delta)a\\
&= \frac{\alpha}{z} + (1-\alpha)(\pi^h u'_h + (1-\pi^h)u'_l) - 2\alpha((1-\eta^h)\pi^h u'_h - \eta^h(1-\pi^h)u'_l)a\\
&+ 2\frac{\alpha}{z}\left(\pi^h((1-\eta^h) - 2(1-\eta^h)^2\varepsilon) + (1-\pi^h)(-\eta^h - 2\eta^{h,2}\varepsilon)\right)(a-\varepsilon).
\end{aligned}$$

Finally, using the fact that $\eta^h = \frac{1-\pi^l}{2-\pi^h-\pi^l}$, we find:

$$\begin{aligned}
\frac{C_1^{ZV,h}}{\beta} &= \frac{\alpha}{z} + (1-\alpha)u'(\delta) - 2\alpha\frac{(1-\pi^h)(\pi^h+\pi^l-1)}{2-\pi^h-\pi^l}u'(\delta)a \\
&+ 2\frac{\alpha}{z}\frac{1-\pi^h}{2-\pi^h-\pi^l}\left(\pi^h+\pi^l-1-2\frac{(1-\pi^h)(\pi^h+\pi^l-1)+(1-\pi^l)(2-\pi^h-\pi^l)}{2-\pi^h-\pi^l}\varepsilon\right)(a-\varepsilon)\\
&= \frac{\alpha}{z} + (1-\alpha)u'(\delta) - 2\alpha\frac{(1-\pi^h)(\pi^h+\pi^l-1)}{2-\pi^h-\pi^l}\left(u'(\delta)-\frac{1}{z}\right)a\\
&- 2\frac{\alpha}{z}\frac{1-\pi^h}{2-\pi^h-\pi^l}\left(\pi^h+\pi^l-1+2\frac{(1-\pi^h)(\pi^h+\pi^l-1)+(1-\pi^l)(2-\pi^h-\pi^l)}{2-\pi^h-\pi^l}(a-\varepsilon)\right)\varepsilon.
\end{aligned} \tag{21}$$

Second-order development of the price $p_1^{ZV,s}$. From Equation (21), we infer that the one-period bond price $p_1^{ZV,h} = C_1^{ZV,k} z^h = C_1^{ZV,h} z(1+2(1-\eta^h)\varepsilon)$ can be expressed as follows:

$$\begin{aligned}
\frac{p_1^{ZV,h}}{\beta} &= \alpha + (1-\alpha)zu'(\delta) - 2\alpha\frac{(1-\pi^h)(\pi^h+\pi^l-1)}{2-\pi^h-\pi^l}(zu'(\delta)-1)a\\
&- 2\alpha\frac{1-\pi^h}{2-\pi^h-\pi^l}\left(\pi^h+\pi^l-1+2\left(\frac{(1-\pi^h)(\pi^h+\pi^l-1)}{2-\pi^h-\pi^l}+1-\pi^l\right)(a-\varepsilon)\right)\varepsilon\\
&+ 2\frac{1-\pi^h}{2-\pi^h-\pi^l}\left(\alpha+(1-\alpha)zu'(\delta)\right)\varepsilon - 4\alpha\frac{(1-\pi^h)^2(\pi^h+\pi^l-1)}{(2-\pi^h-\pi^l)^2}\left((zu'(\delta)-1)a+\varepsilon\right)\varepsilon.
\end{aligned}$$

[1] In what follows, bold numbers for equations refer to equations in the body of the paper.

Gathering terms yields the following expression for the second-order development of $p_1^{ZV,h}$:

$$\begin{aligned}\frac{p_1^{ZV,h}}{\beta} &= \alpha + (1-\alpha)zu'(\delta) \\ &+ 2\frac{1-\pi^h}{2-\pi^h-\pi^l}\left(-\alpha(\pi^h+\pi^l-1)(zu'(\delta)-1)a + \left((2-\pi^h-\pi^l)\alpha + (1-\alpha)zu'(\delta)\right)\varepsilon\right) \\ &- 4\alpha\frac{1-\pi^h}{2-\pi^h-\pi^l}\left(\frac{(1-\pi^h)(\pi^h+\pi^l-1)}{2-\pi^h-\pi^l}zu'(\delta)a + (1-\pi^l)(a-\varepsilon)\right)\varepsilon.\end{aligned} \tag{22}$$

Second-order development of the interest rate $r_1^{ZV,s}$. Using Equation (22), we can express the short yield $r_1^{ZV,h} = -\ln(p_1^{ZV,h})$ as follows:

$$\begin{aligned}r_1^{ZV,h} &= -\ln\left(\beta\alpha + \beta(1-\alpha)zu'(\delta)\right) \\ &- 2\frac{1-\pi^h}{(2-\pi^h-\pi^l)(\alpha+(1-\alpha)zu'(\delta))}\left(-\alpha(\pi^h+\pi^l-1)(zu'(\delta)-1)a + \left((2-\pi^h-\pi^l)\alpha + (1-\alpha)zu'(\delta)\right)\varepsilon\right) \\ &+ 4\alpha\frac{1-\pi^h}{(2-\pi^h-\pi^l)(\alpha+(1-\alpha)zu'(\delta))}\left(\frac{(1-\pi^h)(\pi^h+\pi^l-1)}{2-\pi^h-\pi^l}zu'(\delta)a + (1-\pi^l)(a-\varepsilon)\right)\varepsilon \\ &+ 2\frac{(1-\pi^h)^2}{(2-\pi^h-\pi^l)^2(\alpha+(1-\alpha)zu'(\delta))^2}\left(-\alpha(\pi^h+\pi^l-1)(zu'(\delta)-1)a + \left((2-\pi^h-\pi^l)\alpha + (1-\alpha)zu'(\delta)\right)\varepsilon\right)^2 \\ &= -\ln\left(\beta\alpha + \beta(1-\alpha)zu'(\delta)\right) \\ &- 2\frac{1-\pi^h}{(2-\pi^h-\pi^l)(\alpha+(1-\alpha)zu'(\delta))}\left(-\alpha(\pi^h+\pi^l-1)(zu'(\delta)-1)a + \left((2-\pi^h-\pi^l)\alpha + (1-\alpha)zu'(\delta)\right)\varepsilon\right) \\ &+ 2\frac{\alpha^2(1-\pi^h)^2(\pi^h+\pi^l-1)^2(zu'(\delta)-1)^2}{(2-\pi^h-\pi^l)^2(\alpha+(1-\alpha)zu'(\delta))^2}a^2 \\ &- 4\alpha\frac{(1-\pi^h)^2}{(2-\pi^h-\pi^l)^2(\alpha+(1-\alpha)zu'(\delta))^2}(\pi^h+\pi^l-1)(zu'(\delta)-1)\left((2-\pi^h-\pi^l)\alpha + (1-\alpha)zu'(\delta)\right)a\varepsilon \\ &+ 4\alpha\frac{1-\pi^h}{(2-\pi^h-\pi^l)(\alpha+(1-\alpha)zu'(\delta))}\left(\frac{(1-\pi^h)(\pi^h+\pi^l-1)}{2-\pi^h-\pi^l}zu'(\delta) + 1 - \pi^l\right)a\varepsilon \\ &+ \frac{2(1-\pi^h)}{(2-\pi^h-\pi^l)(\alpha+(1-\alpha)zu'(\delta))}\varepsilon^2\left(-2\alpha(1-\pi^l) + \frac{(1-\pi^h)\left((2-\pi^h-\pi^l)\alpha + (1-\alpha)zu'(\delta)\right)^2}{(2-\pi^h-\pi^l)(\alpha+(1-\alpha)zu'(\delta))}\right)\varepsilon^2.\end{aligned}$$

Finally, we obtain the following expression for the second-order development of $r_1^{ZV,h}$:

$$\begin{aligned}r_1^{ZV,h} &= -\ln\left(\beta\alpha + \beta(1-\alpha)zu'(\delta)\right) \\ &- 2\frac{1-\pi^h}{(2-\pi^h-\pi^l)(\alpha+(1-\alpha)zu'(\delta))}\left(-\alpha(\pi^h+\pi^l-1)(zu'(\delta)-1)a + \left((2-\pi^h-\pi^l)\alpha + (1-\alpha)zu'(\delta)\right)\varepsilon\right) \\ &+ 2\frac{\alpha^2(1-\pi^h)^2(\pi^h+\pi^l-1)^2(zu'(\delta)-1)^2}{(2-\pi^h-\pi^l)^2(\alpha+(1-\alpha)zu'(\delta))^2}a^2 \\ &+ \frac{4\alpha(1-\pi^h)}{(2-\pi^h-\pi^l)(\alpha+(1-\alpha)zu'(\delta))}\left(1-\pi^l + \frac{(1-\pi^h)(\pi^h+\pi^l-1)}{2-\pi^h-\pi^l}\left(1 + \frac{\alpha(\pi^h+\pi^l-1)(zu'(\delta)-1)}{\alpha+(1-\alpha)zu'(\delta)}\right)\right)a\varepsilon \\ &+ \frac{2(1-\pi^h)}{(2-\pi^h-\pi^l)(\alpha+(1-\alpha)zu'(\delta))}\varepsilon^2\left(-2\alpha(1-\pi^l) + \frac{(1-\pi^h)\left((2-\pi^h-\pi^l)\alpha + (1-\alpha)zu'(\delta)\right)^2}{(2-\pi^h-\pi^l)(\alpha+(1-\alpha)zu'(\delta))}\right)\varepsilon^2.\end{aligned} \tag{23}$$

Following the same steps, we obtain the following expression for the second-order development of $r_1^{ZV,l}$:

$$\begin{aligned} r_1^{ZV,l} =& -\ln\left(\beta\alpha+\beta(1-\alpha)zu'(\delta)\right) \\ &+2\frac{1-\pi^l}{(2-\pi^h-\pi^l)(\alpha+(1-\alpha)zu'(\delta))}\left(-\alpha(\pi^h+\pi^l-1)(zu'(\delta)-1)a+\left((2-\pi^h-\pi^l)\alpha+(1-\alpha)zu'(\delta)\right)\varepsilon\right) \\ &+2\frac{\alpha^2(1-\pi^l)^2(\pi^h+\pi^l-1)^2(zu'(\delta)-1)^2}{(2-\pi^h-\pi^l)^2(\alpha+(1-\alpha)zu'(\delta))^2}a^2 \\ &+\frac{4\alpha(1-\pi^l)}{(2-\pi^h-\pi^l)(\alpha+(1-\alpha)zu'(\delta))}\left(1-\pi^h+\frac{(1-\pi^l)(\pi^h+\pi^l-1)}{2-\pi^h-\pi^l}\left(1+\frac{\alpha(\pi^h+\pi^l-1)(zu'(\delta)-1)}{\alpha+(1-\alpha)zu'(\delta)}\right)\right)a\varepsilon \\ &+\frac{2(1-\pi^l)}{(2-\pi^h-\pi^l)(\alpha+(1-\alpha)zu'(\delta))}\varepsilon^2\left(-2\alpha(1-\pi^h)+\frac{(1-\pi^l)\left((2-\pi^h-\pi^l)\alpha+(1-\alpha)zu'(\delta)\right)^2}{(2-\pi^h-\pi^l)(\alpha+(1-\alpha)zu'(\delta))}\right)\varepsilon^2. \end{aligned} \tag{24}$$

Using Equations (23) and (24), we infer the second-order expression for the average short rate, $\bar{r}_1^{ZV}=\frac{1}{2-\pi^h-\pi^l}((1-\pi^l)r_1^{ZV,h}+(1-\pi^h)r_1^{ZV,l})$, to be:

$$\begin{aligned} r_1^{ZV} =& -\ln\left(\beta\alpha+\beta(1-\alpha)zu'(\delta)\right) \\ &+2\frac{\alpha^2(1-\pi^h)(1-\pi^l)(\pi^h+\pi^l-1)^2(zu'(\delta)-1)^2}{(2-\pi^h-\pi^l)^2(\alpha+(1-\alpha)zu'(\delta))^2}a^2 \\ &+\frac{4\alpha(1-\pi^h)(1-\pi^l)}{(2-\pi^h-\pi^l)^2(\alpha+(1-\alpha)zu'(\delta))}\left(1+(\pi^h+\pi^l-1)^2\frac{\alpha(zu'(\delta)-1)}{\alpha+(1-\alpha)zu'(\delta)}\right)a\varepsilon \\ &+\frac{2(1-\pi^h)(1-\pi^l)}{(2-\pi^h-\pi^l)(\alpha+(1-\alpha)zu'(\delta))}\left(-2\alpha+\frac{\left((2-\pi^h-\pi^l)\alpha+(1-\alpha)zu'(\delta)\right)^2}{(2-\pi^h-\pi^l)(\alpha+(1-\alpha)zu'(\delta))}\right)\varepsilon^2. \end{aligned}$$

This expression $\bar{r}_1^{ZV}$ finally simplifies into:

$$\begin{aligned} \bar{r}_1^{ZV} =& -\ln\left(\beta\alpha+\beta(1-\alpha)zu'(\delta)\right) \\ &+2\frac{\alpha^2(1-\pi^h)(1-\pi^l)(\pi^h+\pi^l-1)^2(zu'(\delta)-1)^2}{(2-\pi^h-\pi^l)^2(\alpha+(1-\alpha)zu'(\delta))^2}a^2 \\ &+\frac{4\alpha(1-\pi^h)(1-\pi^l)}{(2-\pi^h-\pi^l)^2(\alpha+(1-\alpha)zu'(\delta))^2}\left(zu'(\delta)-\alpha(2-\pi^h-\pi^l)(\pi^h+\pi^l)(zu'(\delta)-1)\right)a\varepsilon \\ &+\frac{2(1-\pi^h)(1-\pi^l)}{(2-\pi^h-\pi^l)(\alpha+(1-\alpha)zu'(\delta))}\left(-2\alpha+\frac{\left((2-\pi^h-\pi^l)\alpha+(1-\alpha)zu'(\delta)\right)^2}{(2-\pi^h-\pi^l)(\alpha+(1-\alpha)zu'(\delta))}\right)\varepsilon^2, \end{aligned} \tag{25}$$

which is expression (18) in Lemma 2.

1.3.2 The long term rate

From Equation **(38)** in the body of the paper, we can express the long yield r_∞^{ZV} as follows:

$$r_\infty^{ZV} = -\ln\left(\widehat{p}_\infty^{ZV}\right), \tag{26}$$

$$\text{where:}\quad \widehat{p}_\infty^{ZV} = \frac{\beta}{2}\left(\pi^h\kappa^{ZV,h}+\pi^l\kappa^{ZV,l}+\left(\left(\pi^h\kappa^{ZV,h}+\pi^l\kappa^{ZV,l}\right)^2-4(\pi^h+\pi^l-1)\kappa^{ZV,h}\kappa^{ZV,l}\right)^{\frac{1}{2}}\right), \tag{27}$$

$$\text{and}\quad \kappa^{ZV,s} = \alpha^s+(1-\alpha^s)z^s u'(\delta),\quad s=h,l, \tag{28}$$

where $\widehat{p}_\infty^{ZV}\equiv\left(p_\infty^{ZV}\right)^k$.

The computation of the second-order development of r_∞^{ZV} is more cumbersome, so we proceed in several steps to make it more transparent. Namely, we derive the second-order expressions for (i) $\kappa^{ZV,s}$, (ii) $\kappa^{ZV,h}\kappa^{ZV,l}$, (iii) $\left(\pi^h\kappa^{ZV,h}+\pi^l\kappa^{ZV,l}\right)^2$, (iv) $\left(\pi^h\kappa^{ZV,h}+\pi^l\kappa^{ZV,l}\right)^2-4(\pi^h+\pi^l-1)\kappa^{ZV,h}\kappa^{ZV,l}$, (v) $\left(\left(\pi^h\kappa^{ZV,h}+\pi^l\kappa^{ZV,l}\right)^2-4(\pi^h+\pi^l-1)\kappa^{ZV,h}\kappa^{ZV,l}\right)^{\frac{1}{2}}$, (vi) $\widehat{p}_\infty^{ZV}$, and, finally, (vii) r_∞^{ZV}.

Second-order development of $\kappa^{ZV,s}$. Substituting the expressions for z^s and α^s in (16) and (17) into the expression for $\kappa^{ZV,s}$ in (28), we obtain:

$$\kappa^{ZV,h}=\alpha+(1-\alpha)zu'(\delta)+\frac{2(1-\pi^h)}{2-\pi^h-\pi^l}\left(-\alpha(zu'(\delta)-1)a+(1-\alpha)zu'(\delta)\varepsilon-\alpha\frac{2(1-\pi^h)}{2-\pi^h-\pi^l}zu'(\delta)\varepsilon a\right) \tag{29}$$

$$\kappa^{ZV,l}=\alpha+(1-\alpha)zu'(\delta)+\frac{2(1-\pi^l)}{2-\pi^h-\pi^l}\left(\alpha(zu'(\delta)-1)a-(1-\alpha)zu'(\delta)\varepsilon-\alpha\frac{2(1-\pi^l)}{2-\pi^h-\pi^l}zu'(\delta)\varepsilon a\right) \tag{30}$$

Second-order development of $\kappa^{ZV,h}\kappa^{ZV,l}$. From (29)–(30) above, we infer the expression for $\kappa^{ZV,h}\kappa^{ZV,l}$:

$$\begin{aligned}\kappa^{ZV,h}\kappa^{ZV,l}&=(\alpha+(1-\alpha)zu'(\delta))^2\\&+\frac{2(\pi^h-\pi^l)}{2-\pi^h-\pi^l}\left(\alpha+(1-\alpha)zu'(\delta)\right)\left(\alpha(zu'(\delta)-1)a-(1-\alpha)zu'(\delta)\varepsilon\right)\\&+\frac{4\alpha zu'(\delta)}{(2-\pi^h-\pi^l)^2}\Big(2(1-\pi^h)(1-\pi^l)(1-\alpha)(zu'(\delta)-1)\\&\qquad-\left((1-\pi^h)^2+(1-\pi^l)^2\right)(\alpha+(1-\alpha)u'(\delta)z)\Big)\varepsilon a\\&-\frac{4(1-\pi^h)(1-\pi^l)}{(2-\pi^h-\pi^l)^2}\left(\alpha^2(zu'(\delta)-1)^2a^2+(1-\alpha)^2u'(\delta)^2z^2\varepsilon^2\right)\end{aligned}$$

Rearranging this expression, we obtain:

$$\begin{aligned}&\kappa^{ZV,h}\kappa^{ZV,l}=(\alpha+(1-\alpha)zu'(\delta))^2\\&+\frac{2(\pi^h-\pi^l)}{2-\pi^h-\pi^l}\left(\alpha+(1-\alpha)zu'(\delta)\right)\left(\alpha(zu'(\delta)-1)a-(1-\alpha)zu'(\delta)\varepsilon\right)\\&-\frac{4\alpha zu'(\delta)}{(2-\pi^h-\pi^l)^2}\left((\pi^h-\pi^l)^2(1-\alpha)(zu'(\delta)-1)+(1-\pi^h)^2+(1-\pi^l)^2\right)\varepsilon a\\&-\frac{4(1-\pi^h)(1-\pi^l)}{(2-\pi^h-\pi^l)^2}\left(\alpha^2(zu'(\delta)-1)^2a^2+(1-\alpha)^2u'(\delta)^2z^2\varepsilon^2\right)\end{aligned} \tag{31}$$

Second-order development of $\left(\pi^h\kappa^{ZV,h}+\pi^l\kappa^{ZV,l}\right)^2$. Using (29)–(30) again, we infer the expression for $\pi^h\kappa^{ZV,h}+\pi^l\kappa^{ZV,l}$:

$$\begin{aligned}\pi^h\kappa^{ZV,h}+\pi^l\kappa^{ZV,l}&=(\pi^h+\pi^l)(\alpha+(1-\alpha)zu'(\delta))\\&+2\frac{(\pi^h-\pi^l)(\pi^h+\pi^l-1)}{2-\pi^h-\pi^l}\left(\alpha(zu'(\delta)-1)a-(1-\alpha)zu'(\delta)\varepsilon\right)\\&-4\alpha\frac{\pi^h(1-\pi^h)^2+\pi^l(1-\pi^l)^2}{(2-\pi^h-\pi^l)^2}zu'(\delta)\varepsilon a.\end{aligned} \tag{32}$$

Taking the square of the above expression yields:

$$
\begin{aligned}
(\pi^h\kappa^{ZV,h} + \pi^l\kappa^{ZV,l})^2 &= (\pi^h+\pi^l)^2(\alpha+(1-\alpha)zu'(\delta))^2 \\
&+ 4\frac{\left((\pi^h)^2-(\pi^l)^2\right)(\pi^h+\pi^l-1)}{2-\pi^h-\pi^l}(\alpha+(1-\alpha)zu'(\delta))\left(\alpha(zu'(\delta)-1)a-(1-\alpha)zu'(\delta)\varepsilon\right) \\
&- 8\alpha(\pi^h+\pi^l)\frac{\pi^h(1-\pi^h)^2+\pi^l(1-\pi^l)^2}{(2-\pi^h-\pi^l)^2}(\alpha+(1-\alpha)zu'(\delta))zu'(\delta)\varepsilon a \\
&+ 4\frac{(\pi^h-\pi^l)^2(\pi^h+\pi^l-1)^2}{(2-\pi^h-\pi^l)^2}\left(\alpha(zu'(\delta)-1)a-(1-\alpha)zu'(\delta)\varepsilon\right)^2 \\
&= (\pi^h+\pi^l)^2(\alpha+(1-\alpha)zu'(\delta))^2 \\
&+ 4\frac{\left((\pi^h)^2-(\pi^l)^2\right)(\pi^h+\pi^l-1)}{2-\pi^h-\pi^l}(\alpha+(1-\alpha)zu'(\delta))\left(\alpha(zu'(\delta)-1)a-(1-\alpha)zu'(\delta)\varepsilon\right) \\
&+ 4\frac{(\pi^h-\pi^l)^2(\pi^h+\pi^l-1)^2}{(2-\pi^h-\pi^l)^2}\left(\alpha^2(zu'(\delta)-1)^2a^2+(1-\alpha)^2z^2u'(\delta)^2\varepsilon^2\right) \\
&- 8\alpha\Bigg(\frac{(\pi^h-\pi^l)^2(\pi^h+\pi^l-1)^2}{(2-\pi^h-\pi^l)^2}(1-\alpha)(zu'(\delta)-1) \\
&+(\pi^h+\pi^l)\frac{\pi^h(1-\pi^h)^2+\pi^l(1-\pi^l)^2}{(2-\pi^h-\pi^l)^2}(\alpha+(1-\alpha)zu'(\delta))\Bigg)zu'(\delta)\varepsilon a.
\end{aligned}
$$

Rearranging this expression, we obtain:

$$
\begin{aligned}
\left(\pi^h\kappa^{ZV,h} + \pi^l\kappa^{ZV,l}\right)^2 &= (\pi^h+\pi^l)^2(\alpha+(1-\alpha)zu'(\delta))^2 \\
&+ 4\frac{\left((\pi^h)^2-(\pi^l)^2\right)(\pi^h+\pi^l-1)}{2-\pi^h-\pi^l}(\alpha+(1-\alpha)zu'(\delta))\left(\alpha(zu'(\delta)-1)a-(1-\alpha)zu'(\delta)\varepsilon\right) \\
&+ 4\frac{(\pi^h-\pi^l)^2(\pi^h+\pi^l-1)^2}{(2-\pi^h-\pi^l)^2}\left(\alpha^2(zu'(\delta)-1)^2a^2+(1-\alpha)^2z^2u'(\delta)^2\varepsilon^2\right) \\
&- 8\alpha\left(\left(\pi^h\pi^l+\frac{2(\pi^h-\pi^l)^2(\pi^h+\pi^l-1)^2}{(2-\pi^h-\pi^l)^2}\right)(1-\alpha)(zu'(\delta)-1)+(\pi^h+\pi^l)\frac{\pi^h(1-\pi^h)^2+\pi^l(1-\pi^l)^2}{(2-\pi^h-\pi^l)^2}\right)zu'(\delta)\varepsilon a.
\end{aligned} \tag{33}
$$

Second-order development of $\left(\pi^h\kappa^{ZV,h}+\pi^l\kappa^{ZV,l}\right)^2-4(\pi^h+\pi^l-1)\kappa^{ZV,h}\kappa^{ZV,l}$. Using equations (31) and (33) gives the following expression for $\left(\pi^h\kappa^{ZV,h}+\pi^l\kappa^{ZV,l}\right)^2-4(\pi^h+\pi^l-1)\kappa^{ZV,h}\kappa^{ZV,l}$:

$$
\begin{aligned}
&\left(\pi^h\kappa^{ZV,h}+\pi^l\kappa^{ZV,l}\right)^2-4(\pi^h+\pi^l-1)\kappa^{ZV,h}\kappa^{ZV,l} = \left((\pi^h+\pi^l)^2-4(\pi^h+\pi^l-1)\right)(\alpha+(1-\alpha)zu'(\delta))^2 \\
&+ 4\frac{\left((\pi^h)^2-(\pi^l)^2\right)(\pi^h+\pi^l-1)}{2-\pi^h-\pi^l}(\alpha+(1-\alpha)zu'(\delta))\left(\alpha(zu'(\delta)-1)a-(1-\alpha)zu'(\delta)\varepsilon\right) \\
&- 4(\pi^h+\pi^l-1)\frac{2(\pi^h-\pi^l)}{2-\pi^h-\pi^l}\left(\alpha+(1-\alpha)u'(\delta)z\right)\left(\alpha(zu'(\delta)-1)a-(1-\alpha)zu'(\delta)\varepsilon\right) \\
&+ 4\frac{(\pi^h-\pi^l)^2(\pi^h+\pi^l-1)^2}{(2-\pi^h-\pi^l)^2}\left(\alpha^2(zu'(\delta)-1)^2a^2+(1-\alpha)^2z^2u'(\delta)^2\varepsilon^2\right) \\
&+ 4(\pi^h+\pi^l-1)\frac{4(1-\pi^h)(1-\pi^l)}{(2-\pi^h-\pi^l)^2}\left(\alpha^2(zu'(\delta)-1)^2a^2+(1-\alpha)^2u'(\delta)^2z^2\varepsilon^2\right) \\
&- 8\alpha\left(\left(\pi^h\pi^l+\frac{2(\pi^h-\pi^l)^2(\pi^h+\pi^l-1)^2}{(2-\pi^h-\pi^l)^2}\right)(1-\alpha)(zu'(\delta)-1)+(\pi^h+\pi^l)\frac{\pi^h(1-\pi^h)^2+\pi^l(1-\pi^l)^2}{(2-\pi^h-\pi^l)^2}\right)zu'(\delta)\varepsilon a \\
&+ 16\alpha(\pi^h+\pi^l-1)\left(\frac{(\pi^h-\pi^l)^2}{(2-\pi^h-\pi^l)^2}(1-\alpha)(zu'(\delta)-1)+\frac{(1-\pi^h)^2+(1-\pi^l)^2}{(2-\pi^h-\pi^l)^2}\right)zu'(\delta)\varepsilon a.
\end{aligned}
$$

Gathering terms appropriately, we obtain:

$$\left(\pi^h\kappa^{ZV,h}+\pi^l\kappa^{ZV,l}\right)^2-4(\pi^h+\pi^l-1)\kappa^{ZV,h}\kappa^{ZV,l}=(2-\pi^h+\pi^l)^2(\alpha+(1-\alpha)zu'(\delta))^2 \tag{34}$$
$$-4(\pi^h+\pi^l-1)(\pi^h-\pi^l)(\alpha+(1-\alpha)zu'(\delta))\left(\alpha(zu'(\delta)-1)a-(1-\alpha)zu'(\delta)\varepsilon\right)$$
$$+4(\pi^h+\pi^l-1)\left(1-\frac{(\pi^h-\pi^l)^2}{2-\pi^h-\pi^l}\right)\left(\alpha^2(zu'(\delta)-1)^2a^2+(1-\alpha)^2u'(\delta)^2z^2\varepsilon^2\right)$$
$$-8\alpha\left((1-\pi^h)(1-\pi^l)-(\pi^h+\pi^l-1)\frac{(\pi^h-\pi^l)^2}{2-\pi^h-\pi^l}+\left(\pi^h\pi^l-\frac{2(\pi^h+\pi^l-1)(\pi^h-\pi^l)^2}{2-\pi^h-\pi^l}\right)(1-\alpha)(zu'(\delta)-1)\right)zu'(\delta)\varepsilon a.$$

Second-order development of $\left(\left(\pi^h\kappa^{ZV,h}+\pi^l\kappa^{ZV,l}\right)^2-4(\pi^h+\pi^l-1)\kappa^{ZV,h}\kappa^{ZV,l}\right)^{\frac{1}{2}}$. Taking the square root of (34) provides the following expression:

$$\left(\left(\pi^h\kappa^{ZV,h}+\pi^l\kappa^{ZV,l}\right)^2-4(\pi^h+\pi^l-1)\kappa^{ZV,h}\kappa^{ZV,l}\right)^{\frac{1}{2}}=(2-\pi^h+\pi^l)(\alpha+(1-\alpha)zu'(\delta))\times$$
$$\left(1-4\frac{(\pi^h+\pi^l-1)(\pi^h-\pi^l)}{(2-\pi^h+\pi^l)^2(\alpha+(1-\alpha)zu'(\delta))}\left(\alpha(zu'(\delta)-1)a-(1-\alpha)zu'(\delta)\varepsilon\right)\right.$$
$$+4\frac{\pi^h+\pi^l-1}{(2-\pi^h+\pi^l)^2}\left(1-\frac{(\pi^h-\pi^l)^2}{2-\pi^h-\pi^l}\right)\frac{\alpha^2(zu'(\delta)-1)^2a^2+(1-\alpha)^2u'(\delta)^2z^2\varepsilon^2}{(\alpha+(1-\alpha)zu'(\delta))^2}$$
$$\left.-8\alpha\frac{(1-\pi^h)(1-\pi^l)-(\pi^h+\pi^l-1)\frac{(\pi^h-\pi^l)^2}{2-\pi^h-\pi^l}+\left(\pi^h\pi^l-\frac{2(\pi^h+\pi^l-1)(\pi^h-\pi^l)^2}{2-\pi^h-\pi^l}\right)(1-\alpha)(zu'(\delta)-1)}{(2-\pi^h+\pi^l)^2(\alpha+(1-\alpha)zu'(\delta))^2}zu'(\delta)a\varepsilon\right)^{\frac{1}{2}}.$$

Developing the previous equation at the second-order gives:

$$\left(\left(\pi^h\kappa^{ZV,h}+\pi^l\kappa^{ZV,l}\right)^2-4(\pi^h+\pi^l-1)\kappa^{ZV,h}\kappa^{ZV,l}\right)^{\frac{1}{2}}=(2-\pi^h+\pi^l)(\alpha+(1-\alpha)zu'(\delta))\times$$
$$\left(1-2\frac{(\pi^h+\pi^l-1)(\pi^h-\pi^l)}{(2-\pi^h+\pi^l)^2(\alpha+(1-\alpha)zu'(\delta))}\left(\alpha(zu'(\delta)-1)a-(1-\alpha)zu'(\delta)\varepsilon\right)\right.$$
$$+2\frac{\pi^h+\pi^l-1}{(2-\pi^h+\pi^l)^2}\left(1-\frac{(\pi^h-\pi^l)^2}{2-\pi^h-\pi^l}\right)\frac{\alpha^2(zu'(\delta)-1)^2a^2+(1-\alpha)^2u'(\delta)^2z^2\varepsilon^2}{(\alpha+(1-\alpha)zu'(\delta))^2}$$
$$-2\frac{(\pi^h+\pi^l-1)^2(\pi^h-\pi^l)^2}{(2-\pi^h+\pi^l)^4}\frac{\alpha^2(zu'(\delta)-1)^2a^2+(1-\alpha)^2z^2u'(\delta)^2\varepsilon^2}{(\alpha+(1-\alpha)zu'(\delta))^2}$$
$$+4\alpha\frac{(\pi^h+\pi^l-1)^2(\pi^h-\pi^l)^2}{(2-\pi^h+\pi^l)^4(\alpha+(1-\alpha)zu'(\delta))^2}(1-\alpha)(zu'(\delta)-1)zu'(\delta)a\varepsilon$$
$$\left.-4\alpha\frac{(1-\pi^h)(1-\pi^l)-(\pi^h+\pi^l-1)\frac{(\pi^h-\pi^l)^2}{2-\pi^h-\pi^l}+\left(\pi^h\pi^l-\frac{2(\pi^h+\pi^l-1)(\pi^h-\pi^l)^2}{2-\pi^h-\pi^l}\right)(1-\alpha)(zu'(\delta)-1)}{(2-\pi^h+\pi^l)^2(\alpha+(1-\alpha)zu'(\delta))^2}zu'(\delta)a\varepsilon\right).$$

The expression $\left(\left(\pi^h\kappa^{ZV,h}+\pi^l\kappa^{ZV,l}\right)^2-4(\pi^h+\pi^l-1)\kappa^{ZV,h}\kappa^{ZV,l}\right)^{\frac{1}{2}}$ becomes, after gathering properly terms in the previous equation:

$$\left(\left(\pi^h\kappa^{ZV,h}+\pi^l\kappa^{ZV,l}\right)^2-4(\pi^h+\pi^l-1)\kappa^{ZV,h}\kappa^{ZV,l}\right)^{\frac{1}{2}}=(2-\pi^h+\pi^l)(\alpha+(1-\alpha)zu'(\delta)) \tag{35}$$
$$-2\frac{(\pi^h+\pi^l-1)(\pi^h-\pi^l)}{(2-\pi^h+\pi^l)}\left(\alpha(zu'(\delta)-1)a-(1-\alpha)zu'(\delta)\varepsilon\right)$$
$$+8\frac{(\pi^h+\pi^l-1)(1-\pi^h)(1-\pi^l)}{(2-\pi^h+\pi^l)^3}\frac{\alpha^2(zu'(\delta)-1)^2a^2+(1-\alpha)^2u'(\delta)^2z^2\varepsilon^2}{(\alpha+(1-\alpha)zu'(\delta))}$$
$$+4\alpha\left(\frac{(\pi^h+\pi^l-1)(\pi^h-\pi^l)^2(3-\pi^h+\pi^l)}{(2-\pi^h+\pi^l)^2}-\pi^h\pi^l\right)\frac{(1-\alpha)(zu'(\delta)-1)}{(2-\pi^h+\pi^l)(\alpha+(1-\alpha)zu'(\delta))}zu'(\delta)a\varepsilon$$
$$+4\alpha\frac{(\pi^h+\pi^l-1)\frac{(\pi^h-\pi^l)^2}{2-\pi^h-\pi^l}-(1-\pi^h)(1-\pi^l)}{(2-\pi^h+\pi^l)(\alpha+(1-\alpha)zu'(\delta))}zu'(\delta)a\varepsilon.$$

Second-order development of $\widehat{p}_\infty^{ZV}$. Using (32) and (35), we infer the second-order expression for $\widehat{p}_\infty^{ZV}$ defined in equation (27):

$$\frac{2\widehat{p}_\infty^{ZV}}{\beta}=(2-\pi^h+\pi^l)(\alpha+(1-\alpha)zu'(\delta))+(\pi^h+\pi^l)(\alpha+(1-\alpha)zu'(\delta))$$
$$+8\frac{(\pi^h+\pi^l-1)(1-\pi^h)(1-\pi^l)}{(2-\pi^h+\pi^l)^3}\frac{\alpha^2(zu'(\delta)-1)^2a^2+(1-\alpha)^2u'(\delta)^2z^2\varepsilon^2}{(\alpha+(1-\alpha)zu'(\delta))}$$
$$+4\alpha\left(\frac{(\pi^h+\pi^l-1)(\pi^h-\pi^l)^2(3-\pi^h+\pi^l)}{(2-\pi^h+\pi^l)^2}-\pi^h\pi^l\right)\frac{(1-\alpha)(zu'(\delta)-1)}{(2-\pi^h+\pi^l)(\alpha+(1-\alpha)zu'(\delta))}zu'(\delta)a\varepsilon$$
$$+4\alpha\frac{(\pi^h+\pi^l-1)\frac{(\pi^h-\pi^l)^2}{2-\pi^h-\pi^l}-(1-\pi^h)(1-\pi^l)}{(2-\pi^h+\pi^l)(\alpha+(1-\alpha)zu'(\delta))}zu'(\delta)a\varepsilon$$
$$-4\alpha\frac{\pi^h(1-\pi^h)^2+\pi^l(1-\pi^l)^2}{2-\pi^h-\pi^l}\frac{1+(1-\alpha)(zu'(\delta)-1)}{(2-\pi^h+\pi^l)(\alpha+(1-\alpha)zu'(\delta))}zu'(\delta)\varepsilon a.$$

Finally, the second-order expression for $\widehat{p}_\infty^{ZV}$ can be expressed as follows:

$$\frac{\widehat{p}_\infty^{ZV}}{\beta}=(\alpha+(1-\alpha)zu'(\delta)) \tag{36}$$
$$+4\frac{(\pi^h+\pi^l-1)(1-\pi^h)(1-\pi^l)}{(2-\pi^h+\pi^l)^3}\frac{\alpha^2(zu'(\delta)-1)^2a^2+(1-\alpha)^2u'(\delta)^2z^2\varepsilon^2}{(\alpha+(1-\alpha)zu'(\delta))}$$
$$-4\alpha\frac{(1-\pi^h)(1-\pi^l)zu'(\delta)}{(2-\pi^h+\pi^l)^2(\alpha+(1-\alpha)zu'(\delta))}\left(1+\frac{\pi^h+\pi^l}{2-\pi^h-\pi^l}(1-\alpha)(zu'(\delta)-1)\right)a\varepsilon.$$

Second-order development of the long-run interest rate r_∞^{ZV}. Using the expression for $\widehat{p}_\infty^{ZV}$ in (36), we can compute the second-order expression for the long yield $r_\infty^{ZV}=-\ln(\widehat{p}_\infty^{ZV})$ (see Eq. (26)):

$$r_\infty^{ZV}=-\ln\left(\beta(\alpha+(1-\alpha)zu'(\delta))\right) \tag{37}$$
$$-4\frac{(\pi^h+\pi^l-1)(1-\pi^h)(1-\pi^l)}{(2-\pi^h+\pi^l)^3}\frac{\alpha^2(zu'(\delta)-1)^2a^2+(1-\alpha)^2u'(\delta)^2z^2\varepsilon^2}{(\alpha+(1-\alpha)zu'(\delta))^2}$$
$$+4\alpha\frac{(1-\pi^h)(1-\pi^l)zu'(\delta)}{(2-\pi^h+\pi^l)^2(\alpha+(1-\alpha)zu'(\delta))^2}\left(1+\frac{\pi^h+\pi^l}{2-\pi^h-\pi^l}(1-\alpha)(zu'(\delta)-1)\right)a\varepsilon,$$

which proves (19) in Lemma 2.

Second-order development of the unconditional slope Δ^{ZV}. Using the expression for $\bar{r}_1^{ZV}$ in (25) together with that for r_∞^{ZV} in (37), we find the following unconditional slope, $\Delta^{ZV}=$

$r_{\infty}^{ZV} - \bar{r}_1^{ZV}$:

$$\Delta^{ZV} = \frac{-2(1-\pi^h)(1-\pi^l)}{(2-\pi^h+\pi^l)^3(\alpha+(1-\alpha)zu'(\delta))^2}\Big(2(\pi^h+\pi^l-1)(1-\alpha)^2u'(\delta)^2z^2$$
$$-2\alpha(\alpha+(1-\alpha)zu'(\delta))(2-\pi^h-\pi^l)^2+(2-\pi^h-\pi^l)\Big((2-\pi^h-\pi^l)\alpha+(1-\alpha)zu'(\delta)\Big)^2\Big)\varepsilon^2$$
$$-2\frac{(1-\pi^h)(1-\pi^l)(\pi^h+\pi^l-1)}{(2-\pi^h+\pi^l)^3(\alpha+(1-\alpha)zu'(\delta))^2}\alpha^2(zu'(\delta)-1)^2a^2\Big(2+(\pi^h+\pi^l-1)^2(2-\pi^h-\pi^l)\Big)a^2$$
$$+\frac{4\alpha(1-\pi^h)(1-\pi^l)}{(2-\pi^h+\pi^l)^3(\alpha+(1-\alpha)zu'(\delta))^2}\Big((2-\pi^h-\pi^l)zu'(\delta)+zu'(\delta)(\pi^h+\pi^l)(1-\alpha)(zu'(\delta)-1)\Big)a\varepsilon$$
$$-\frac{4\alpha(1-\pi^h)(1-\pi^l)}{(2-\pi^h-\pi^l)^3(\alpha+(1-\alpha)zu'(\delta))^2}\Big((2-\pi^h-\pi^l)zu'(\delta)-\alpha(2-\pi^h-\pi^l)^2(\pi^h+\pi^l)(zu'(\delta)-1)\Big)a\varepsilon.$$

Gathering properly the different terms provides the following expression:

$$\Delta^{ZV} = \frac{2(1-\pi^h)(1-\pi^l)(\pi^h+\pi^l)}{(2-\pi^h+\pi^l)^3(\alpha+(1-\alpha)zu'(\delta))^2}\Big((2-\pi^h-\pi^l)^2\alpha^2-(1-\alpha)^2u'(\delta)^2z^2\Big)\varepsilon^2 \tag{38}$$
$$-\frac{2(\pi^h+\pi^l-1)(1-\pi^h)(1-\pi^l)(\pi^h+\pi^l)(3-\pi^h+\pi^l)}{(2-\pi^h+\pi^l)^3}\frac{\alpha^2(zu'(\delta)-1)^2}{(\alpha+(1-\alpha)zu'(\delta))^2}a^2$$
$$+\frac{4\alpha(1-\pi^h)(1-\pi^l)(\pi^h+\pi^l)}{(2-\pi^h+\pi^l)^3(\alpha+(1-\alpha)zu'(\delta))^2}\Big(\alpha(2-\pi^h-\pi^l)^2+zu'(\delta)(1-\alpha)\Big)(zu'(\delta)-1)a\varepsilon,$$

which proves expression (20) of Lemma 2 and concludes the lemma's proof.

1.4 Proof of Proposition 3

We start with a preliminary lemma.

Lemma 3 *Consider the following mean-preserving spread in aggregate and idiosyncratic risks around their unconditional means* (z,α)*:* $\alpha^h = \alpha + 2\eta^l a$*,* $\alpha^l = \alpha - 2\left(1-\eta^l\right)a$*,* $z^h = z\left(1+2\eta^l\varepsilon\right)$ *and* $z^l = z\left(1-2\left(1-\eta^l\right)\varepsilon\right)$ *(where* $\eta^l = \left(1-\pi^l\right)/\left(2-\pi^l-\pi^h\right)$*. A second-order development in* a *and* ε *gives*

$$\Delta^{ZV} = \frac{2(1-\pi^h)(1-\pi^l)\Sigma\pi\Omega^2}{(2-\Sigma\pi)}\big(-(3-\Sigma\pi)(\Sigma\pi-1)(zu'(\delta)-1)^2a^2$$
$$+2(zu'(\delta)-1)(\alpha(2-\Sigma\pi)^2+(1-\alpha)zu'(\delta))\varepsilon\, a+\left(\alpha^2(2-\Sigma\pi)^2-(1-\alpha)^2z^2u'(\delta)^2\right)\varepsilon^2\big),$$

where for sake of clarity, we define $\Sigma\pi \equiv \pi^l+\pi^h$*, and* $\Omega \equiv [(2-\Sigma\pi)(\alpha+(1-\alpha)zu'(\delta))]^{-1}$*.*

Proof. It follows directly from our preliminary result and Lemma 2. □

We may now turn to the core of the proof of Proposition 3.

1. Yield curve without aggregate shocks. Without aggregate shocks, $z^h = z^l = z$ and $\alpha^h = \alpha^l = \alpha$. From (8), the price p_k of a $k-$period bond is simply equal to ($p_0 = 1$):

$$p_k = \beta\left(\alpha+(1-\alpha)zu'(\delta)\right)p_{k-1} = \beta^k\left(\alpha+(1-\alpha)zu'(\delta)\right)^k,$$

which gives $r_k = -\frac{1}{k}\ln(p_k) = -\ln(\beta) - \ln(\alpha + (1-\alpha)zu'(\delta))$, $k = 1, \ldots, n$.

2. Impact of α on the slope. From Lemma 3, when $a = 0$, the second-order expression for the slope simplifies to:

$$\Delta^{ZV} = \frac{2(1-\pi^h)(1-\pi^l)\Sigma\pi}{(2-\Sigma\pi)^3}\frac{\alpha^2(2-\Sigma\pi)^2 - (1-\alpha)^2z^2u'(\delta)^2}{(\alpha + (1-\alpha)zu'(\delta))^2}\varepsilon^2.$$

Taking the derivative of the latter expression with respect to α yields:

$$\begin{aligned}\Delta^{ZV} &= \frac{4(1-\pi^h)(1-\pi^l)\Sigma\pi}{(2-\Sigma\pi)^3\,(\alpha + (1-\alpha)zu'(\delta))^3}\left(\left(\alpha(2-\Sigma\pi)^2 + (1-\alpha)z^2u'(\delta)^2\right)\left(\alpha + (1-\alpha)zu'(\delta)\right)\right.\\ &\quad \left. -\left(\alpha^2(2-\Sigma\pi)^2 - (1-\alpha)^2z^2u'(\delta)^2\right)(1 - zu'(\delta))\right)\varepsilon^2\\ &= 4(1-\pi^h)(1-\pi^l)\Sigma\pi\Omega^3\left(\alpha^2(2-\Sigma\pi)^2zu'(\delta) + \alpha(2-\Sigma\pi)^2(1-\alpha)zu'(\delta)\right.\\ &\quad \left. +(1-\alpha)^2z^2u'(\delta)^2 + \alpha(1-\alpha)z^2u'(\delta)^2\right)\varepsilon^2\\ &= 4(1-\pi^h)(1-\pi^l)\Sigma\pi\Omega^3\left(\alpha(2-\Sigma\pi)^2 + (1-\alpha)zu'(\delta)\right)zu'(\delta)\varepsilon^2 > 0.\end{aligned}$$

Hence, the slope of the yield curve increases with α.

From (24) and (37), we infer the expression for the conditional slope $\Delta^{ZV,l} = r_\infty^{ZV} - r_1^{ZV,l}$ in state l:

$$\begin{aligned}\Delta^{ZV,l} &= -2\frac{1-\pi^l}{(2-\pi^h-\pi^l)(\alpha + (1-\alpha)zu'(\delta))}\Big((2-\pi^h-\pi^l)\alpha + (1-\alpha)zu'(\delta)\Big)\varepsilon\\ &= -2(1-\pi^l)\left(1 + \frac{(\pi^h+\pi^l-1)(1-\alpha)zu'(\delta)}{(2-\pi^h-\pi^l)(\alpha + (1-\alpha)zu'(\delta))}\right)\varepsilon\end{aligned}$$

We deduce that the derivative of the conditional slope can be expressed as follows:

$$\frac{\partial\Delta^{ZV,l}}{\partial\alpha} = 2\frac{(1-\pi^l)(\pi^h+\pi^l-1)}{2-\pi^h-\pi^l}\frac{zu'(\delta)}{(\alpha + (1-\alpha)zu'(\delta))^2}\varepsilon \tag{39}$$

3. Greater slope under incomplete markets/zero-volume than under complete markets From Lemma 3, the slope is a quadratic, concave function of a (the coefficient in a^2 is negative) that admits a unique maximum at a_{max}, defined as $\left.\frac{\partial\Delta^{ZV}}{\partial a}\right|_{a=a_{max}} = 0$. We obtain:

$$a_{max} = \frac{\alpha(2-\Sigma\pi)^2 + (1-\alpha)zu'(\delta)}{(3-\Sigma\pi)(\Sigma\pi-1)(zu'(\delta)-1)}\varepsilon.$$

We deduce the following expression for the upper bound on the slope, Δ^{ZV}_{max} :

$$\begin{aligned}&\Delta^{ZV}_{max} = \frac{2(1-\pi^h)(1-\pi^l)\Sigma\pi\Omega^2}{(2-\Sigma\pi)(3-\Sigma\pi)(\Sigma\pi-1)}K,\\ &\text{with}\quad K = (\alpha(2-\Sigma\pi)^2 + (1-\alpha)zu'(\delta))^2 + (3-\Sigma\pi)(\Sigma\pi-1)\left(\alpha^2(2-\Sigma\pi)^2 - (1-\alpha)^2z^2u'(\delta)^2\right).\end{aligned}$$

We can simplify the expression for K as follows:

$$\begin{aligned}K &= \alpha^2(2-\Sigma\pi)^4 + (1-\alpha)^2 z^2 u'(\delta)^2 + 2(2-\Sigma\pi)^2\alpha(1-\alpha)zu'(\delta)\\&+ \alpha^2(2-\Sigma\pi)^2(3-\Sigma\pi)(\Sigma\pi-1) - (3-\Sigma\pi)(\Sigma\pi-1)(1-\alpha)^2 z^2 u'(\delta)^2\\&= (1-\alpha)^2 z^2 u'(\delta)^2\left(1+3-4\Sigma\pi+\Sigma\pi^2\right) + 2(2-\Sigma\pi)^2\alpha(1-\alpha)zu'(\delta)\\&+ \alpha^2(2-\Sigma\pi)^2\left(4-4\Sigma\pi+\Sigma\pi^2-3+4\Sigma\pi-\Sigma\pi^2\right)\\&= (1-\alpha)^2 z^2 u'(\delta)^2(2-\Sigma\pi)^2 + 2(2-\Sigma\pi)^2\alpha(1-\alpha)zu'(\delta) + \alpha^2(2-\Sigma\pi)^2\\&= (2-\Sigma\pi)^2\left(\alpha+(1-\alpha)zu'(\delta)\right)^2,\end{aligned}$$

which implies that

$$\Delta^{ZV}_{max} = \frac{2(1-\pi^h)(1-\pi^l)\Sigma\pi}{(2-\Sigma\pi)(3-\Sigma\pi)(\Sigma\pi-1)}\varepsilon^2.$$

In order to compare Δ^{ZV}_{max} to the complete-market slope Δ^{CM}, we need to compute that slope. From Lemma 3, we obtain it with $a=0$ and $\alpha=1$ (no idiosyncratic risk), which gives:

$$\Delta^{CM} = \frac{2(1-\pi^h)(1-\pi^l)\Sigma\pi}{2-\Sigma\pi}\varepsilon^2.$$

From the latter two expressions, we find that $\Delta^{ZV}_{max} - \Delta^{CM}$:

$$\begin{aligned}\Delta^{ZV}_{max} - \Delta^{CM} &= \frac{2(1-\pi^h)(1-\pi^l)\Sigma\pi}{(2-\Sigma\pi)(3-\Sigma\pi)(\Sigma\pi-1)}\left(1-(3-\Sigma\pi)(\Sigma\pi-1)\right)\varepsilon^2\\&= \frac{2(2-\Sigma\pi)(1-\pi^h)(1-\pi^l)\Sigma\pi}{(3-\Sigma\pi)(\Sigma\pi-1)}\varepsilon^2 > 0,\end{aligned}$$

Hence, there are values of a such that the slope of the yield curve is larger in the incomplete-market, zero net supply case than in the complete-market case.

1.5 Proof of Proposition 4

1. Equilibrium portfolios. We proceed by construction: we first conjecture, and then derive a sufficient condition for, the existence of an equilibrium in which employed agents are never borrowing-constrained and hold symmetric portfolios, while unemployed are always borrowing-constrained and hold no bonds. Formally, we conjecture:

$$e^i_t = 1 \Rightarrow \varphi^i_{t,k} = 0 \quad \text{and} \quad e^i_t = 0 \Rightarrow \varphi^i_{t,k} > 0 \quad \text{for } k = 1,\dots,n. \tag{40}$$

We remind the agent's budget constraint and the government's budget constraint:

$$c^i_t + \tau_t e^i_t + \sum_{k=1}^{n} p_{t,k}\, b^i_{t,k} = \sum_{k=1}^{n} p_{t,k-1}\, b^i_{t-1,k} + e^i_t z_t l^i_t + \left(1-e^i_t\right)\delta, \tag{41}$$

$$\sum_{k=1}^{n} p_{t,k}\left(s^t\right) A_{t,k}\left(s^t\right) + \omega^e_t\left(s^t\right)\tau_t\left(s^t\right) = \sum_{k=1}^{n} A_{t-k,k}\left(s^{t-1}\right). \tag{42}$$

Conjectured consumption levels and equilibrium pricing kernel. Consider first the consumption level of an unemployed agent at date t. If the agent was employed at date $t-1$, then from the budget constraint (41) and conjecture (40) the agent liquidates his entire portfolio and consume

$$c_t^i = \delta + \sum_{k=1}^{n} p_{t,k-1}\, b_{t-1,k}^i \ (> 0)\,. \tag{43}$$

If, however, this agent was already unemployed at date $t-1$, then by (41) and (40) the agents consumes $c_t^{uu} = \delta > 0$.

Now consider the consumption level of an employed agent at date t. From the FOC of employed agents, this agent consumes $c_t^e = u'^{-1}\left(1/z_t\right)\ (> 0)$ regardless of $e^{i,t}$.

If an employed agent moves into unemployment next period, then his marginal utility will be $u'\left(c_{t+1}^i\right)$, where by construction c_{t+1}^i is given by (43). Then, substituting these marginal utilities into the FOC under conjecture (40), we obtain the Euler equations of employed agents ($k = 1, \ldots, n$):

$$\frac{p_{t,k}}{z_t} = \alpha\beta E_t\left[\frac{p_{t+1,k-1}}{z_{t+1}}\right] + (1-\alpha)\,\beta E_t\left[u'\left(\delta + \sum_{j=1}^{n} p_{t+1,j-1} b_{t,j}^i\right) p_{t+1,k-1}\right]. \tag{44}$$

From (44), the bond demands $b_{t,j}^i$ are functions of aggregate variables only. Total supply being B_k, market clearing requires that $b_{t,k} = B_k/\omega^e$, meaning that no agent holds negative bond quantities. Substituting it in (44) together with (42), we express prices as a function of aggregate variables only.

Following the same steps, borrowing constraint condition (40) becomes:

$$p_{t,k}\, u'\ (\delta) > \beta\,(1-\rho)\, E_t\left[\frac{p_{t+1,k-1}}{z_{t+1}}\right] + \beta\rho E_t\left[p_{t+1,k-1} u'\left(\delta\right)\right], \tag{45}$$

On the other hand, agents who were employed at date $t-1$ and who become unemployed at date t face a binding borrowing constraint iff, for all $k = 1, \ldots, n$:

$$p_{t,k}\, u'\left(\delta + \sum_{j=1}^{n} \frac{p_{t,j-1} B_j}{\omega^e}\right) > \beta(1-\rho) E_t\left[\frac{p_{t+1,k-1}}{z_{t+1}}\right] + \beta\rho E_t\left[p_{t+1,k-1} u'\left(\delta\right)\right]. \tag{46}$$

Since (46) implies (45), we only need to check that the equilibrium satisfies (46).

We prove the existence of the equilibrium in three steps. First, we appropriately set initial conditions so that there is no transitory adjustments in the cross-sectional wealth distribution. Second, we show that the equilibrium exists under zero net bond supply and no aggregate shocks. Third, we show by a continuity argument that the equilibrium exists when volumes and aggregate shocks are small. The technical part is the proof of continuity.

Conditions on agents' initial wealth. We assume that employed agents enter period 0 holding a quantity of bonds $b_{-1,k} = B_k/\omega^e$ with probability α, and no bond with probability $1-\alpha$. Unemployed agents hold no bond with probability ρ, and $b_{-1,k} = B_k/\omega^e$ units of bonds of maturity k with probability $1-\rho$. The initial joint distribution of employment status and bond holdings is thus identical to the stationary distribution.

Existence of a no-trade equilibrium without aggregate shocks. If assets are in zero net supply, then there is no trade between agents and both the liquidation value of the portfolio and taxes will equal zero. Without aggregate uncertainty $z^h = z^l = z$, one easily finds the price of a one period bond $p = m^{ZV}$, where m^{ZV} is given by (47).

$$m_{t+1}^{ZV} = m_{t+1}^{CM} I_{t+1}^{ZV}, \quad \text{with} \quad I_{t+1}^{ZV} = \frac{\alpha_{t+1}/z_{t+1} + (1-\alpha_{t+1})\ u'(\delta)}{1/z_{t+1}} (\geq 1). \tag{47}$$

Rearranging (46) yields the following inequality: $(\alpha + (1-\alpha) z u'(\delta))\, z u'(\delta) > 1 - \rho + \rho\, z u'(\delta)$. Since $zu'(\delta) > 1$ by Assumption B, the right hand side is maximum at $\rho = 1$, in which case the inequality remains true for any value of α; hence the no-trade equilibrium exists in the economy with zero volume and without aggregate risk.

Continuity of the yield curve w.r.t. bond supplies and aggregate shocks. Let us introduce the following change of variables, which greatly simplifies the algebra:

$$C_k^s = p_k^s / z^s, \quad s = h, l, \quad k = 1, \ldots, n. \tag{48}$$

Solving for C_k^s is equivalent to solving for prices (given the z^ss). We now define $B \equiv [B_n \ldots B_1]^\top$ as the vector of bond quantities for the n maturities, $Z \equiv \begin{bmatrix} z^l & z^h \end{bmatrix}^\top$ as the vector of productivity levels, and $C \equiv \begin{bmatrix} C_n^h & C_n^l \ldots C_0^h & C_0^l \end{bmatrix}^\top$ as the vector of price coefficients. 1_n and 0_n are vectors of length n containing respectively only ones and zeros. We then have the following Lemma:

Lemma 4 (Equilibrium existence) *There are neighborhoods $\mathcal{B}$ of 0_n and $\mathcal{Z}$ of 1_2, such that if $B \in \mathcal{B}$ and $Z \in \mathcal{Z}$ then C is a $\mathcal{C}^1$ function of B and of Z.*

Proof. Let us first define $X \equiv [z^h \quad z^l \quad B^\top]$ and:

$$v^s \equiv v\left(\delta + \left(z^h/\omega^e\right) \sum_{j=1}^{n} C_{j-1}^s B_j\right), \tag{49}$$

whether $v = u'$ or u'' (for example, $u'^h \equiv u'(\delta + \left(z^h/\omega^e\right) \sum_{j=1}^n C_{j-1}^h B_j))$. Finally, let $1_{cond.}$ be the function that takes value 1 when *cond.* is true and 0 otherwise, and

$$M(C,X) \equiv \beta \begin{bmatrix} \pi^h(\alpha + (1-\alpha)\, z^h\, u'^h) & (1-\pi^h)(\alpha + (1-\alpha)\, z^l\, u'^l) \\ (1-\pi^l)(\alpha + (1-\alpha)\, z^h\, u'^h) & \pi^l(\alpha + (1-\alpha)\, z^l\, u'^l) \end{bmatrix}. \tag{50}$$

Since $b^i_j = B_j/\omega^e$, (44) can be written as follows:

$$\begin{bmatrix} C^h_k & C^l_k \end{bmatrix}^\top = M(C,X) \cdot \begin{bmatrix} C^h_{k-1} & C^l_{k-1} \end{bmatrix}^\top \text{ for } k = 1, \dots, n. \tag{51}$$

By stacking equalities, we rewrite (51) as $f(C,X) = 0_{(2n+2)\times 1}$, where f is:

$$f(C,X) \equiv C - \begin{bmatrix} 0_{2\times 2} & M(C,X) & 0_{2\times 2} & \dots & 0_{2\times 2} \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & & M(C,X) \\ 0_{2\times 2} & & \dots & & 0_{2\times 2} \end{bmatrix} C - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1/z^h \\ 1/z^l \end{bmatrix}.$$

Since u' is $\mathcal{C}^1$ on $\mathbb{R}$, M and f are also $\mathcal{C}^1$ in (C,X) on $\mathbb{R}^{2n+2} \times \mathbb{R}^{n+2}$. Before using the implicit function theorem to show that C is $\mathcal{C}^1$ in X, we prove that the Jacobian $Df_Y = (\frac{\partial f}{\partial C^h_n}, \frac{\partial f}{\partial C^l_n}, \dots, \frac{\partial f}{\partial C^h_{n-i}}, \frac{\partial f}{\partial C^l_{n-i}}, \dots, \frac{\partial f}{\partial C^h_0}, \frac{\partial f}{\partial C^l_0})$ of f relative to C is invertible at the point $X = (1, 1, 0_{1\times n})$.

1. Derivative of f. We consider the partial derivatives of f relative to C^s_{n-i} ($i = 0, \dots, n$ and $s = h, l$):

$$\frac{\partial f}{\partial C^s_{n-i}} = \frac{\partial C}{\partial C^s_{n-i}} - \mathcal{M}(C,X)\frac{\partial C}{\partial C^s_{n-i}} - \frac{\partial \mathcal{M}(C,X)}{\partial C^s_{n-i}}\, C. \tag{52}$$

To compute the derivative of f, we proceed in 6 steps:

1. we take the derivative of the vector C with respect to C^s_{n-i} and infer the expressions for $\frac{\partial C}{\partial C^s_{n-i}} - \mathcal{M}(C,X)\frac{\partial C}{\partial C^s_{n-i}}$, for $i > 0$;
2. we take the derivative of the vector C and infer the expression for $\frac{\partial C}{\partial C^s_n} - \mathcal{M}(C,X)\frac{\partial C}{\partial C^s_n}$;
3. we take the derivative of the matrix $M(C,X)$ with respect to C^s_{n-i}, for $i > 0$;
4. we take the derivative of the matrix $M(C,X)$ with respect to to C^s_n;
5. we substitute the latter expressions into that for the derivative of f in (52), for $i > 0$;
6. we subsitute the latter expressions into that for the derivative of f in (52), for $i = 0$.

a. Derivative of the vector C and expression for $\frac{\partial C}{\partial C^s_{n-i}} - \mathcal{M}(C,X)\frac{\partial C}{\partial C^s_{n-i}}$, for $i > 0$. We start by taking the derivative of C with respect to C^s_{n-i}, for $i = 0, \ldots, n$ and $s = h, l$:

$$\frac{\partial C}{\partial C^h_{n-i}} = \begin{bmatrix} 0 \;\ldots\; 0 & \underbrace{1}_{rank\ 2i+1} & 0 \;\ldots\; 0 \end{bmatrix}^\top \quad \text{and} \quad \frac{\partial C}{\partial C^l_{n-i}} = \begin{bmatrix} 0 \;\ldots\; 0 & \underbrace{1}_{rank\ 2i+2} & 0 \;\ldots\; 0 \end{bmatrix}^\top .$$

We infer the expression for $\frac{\partial C}{\partial C^h_{n-i}} - \mathcal{M}(C,X)\frac{\partial C}{\partial C^h_{n-i}}$, for $i > 0$:

$$\begin{aligned}
\frac{\partial C}{\partial C^h_{n-i}} - \mathcal{M}(C,X)\frac{\partial C}{\partial C^h_{n-i}} &= \begin{bmatrix} 0_{2i\times 1} \\ 1 \\ 0 \\ 0_{2(n-i)\times 1} \end{bmatrix} \\
&- \beta \begin{bmatrix} 0_{2\times 2} & \begin{bmatrix} \pi^h\left(\alpha+(1-\alpha)\, z^h\, u'^h\right) & (1-\pi^h)\left(\alpha+(1-\alpha)\, z^l\, u'^l\right) \\ (1-\pi^l)\left(\alpha+(1-\alpha)\, z^h\, u'^h\right) & \pi^l\left(\alpha+(1-\alpha)\, z^l\, u'^l\right) \end{bmatrix} & 0_{2\times 2} & 0_{2\times 2} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \begin{bmatrix} \pi^h\left(\alpha+(1-\alpha)\, z^h\, u'^h\right) & (1-\pi^h)\left(\alpha+(1-\alpha)\, z^l\, u'^l\right) \\ (1-\pi^l)\left(\alpha+(1-\alpha)\, z^h\, u'^h\right) & \pi^l\left(\alpha+(1-\alpha)\, z^l\, u'^l\right) \end{bmatrix} \\ 0_{2\times 2} & & \cdots & 0_{2\times 2} \end{bmatrix} \begin{bmatrix} 0_{2i\times 1} \\ 1 \\ 0 \\ 0_{2(n-i)\times 1} \end{bmatrix} \\
&= \begin{bmatrix} 0_{2(i-1)\times 1} \\ -\beta\,(\alpha + (1-\alpha) z^h u'^h)\,\pi^h \\ -\beta\,(\alpha + (1-\alpha) z^h u'^h)\,(1-\pi^l) \\ 1 \\ 0 \\ 0_{2(n-i)\times 1} \end{bmatrix} \begin{matrix} \\ \leftarrow \text{ rank } 2i-1 \\ \\ \leftarrow \text{ rank } 2i+1 \\ \\ \\ \end{matrix} .
\end{aligned}$$

In the same manner, we obtain the expression for $\frac{\partial C}{\partial C^l_{n-i}} - \mathcal{M}(C,X)\frac{\partial C}{\partial C^l_{n-i}}$, for $i > 0$:

$$\frac{\partial C}{\partial C^l_{n-i}} - \mathcal{M}(C,X)\frac{\partial C}{\partial C^l_{n-i}} = \begin{bmatrix} 0_{2(i-1)\times 1} \\ -\beta\,(\alpha + (1-\alpha) z^l u'^l)\,(1-\pi^h) \\ -\beta\,(\alpha + (1-\alpha) z^l u'^l)\,\pi^l \\ 0 \\ 1 \\ 0_{2(n-i)\times 1} \end{bmatrix} \begin{matrix} \\ \leftarrow \text{ rank } 2i-1 \\ \\ \\ \leftarrow \text{ rank } 2i+2 \\ \\ \end{matrix} .$$

b. Derivative of the vector C and expression of $\frac{\partial C}{\partial C^s_n} - \mathcal{M}(C,X)\frac{\partial C}{\partial C^s_n}$. Since the product $\mathcal{M}(C,X)\frac{\partial C}{\partial C^s_n}$ is null, the previous equalities have simple expressions, which are:

$$\frac{\partial C}{\partial C^h_n} - \mathcal{M}(C,X)\frac{\partial C}{\partial C^h_n} = [1\ 0\ \ldots\ 0]^\top \quad \text{and} \quad \frac{\partial C}{\partial C^l_n} - \mathcal{M}(C,X)\frac{\partial C}{\partial C^l_n} = [0\ 1\ 0 \ldots\ 0]^\top .$$

c. Derivative of the matrix $M(C,X)$ with respect to C^s_{n-i}, for $i > 0$. We take the derivative of $M\,(C,X)$ with respect to C^s_j, for $j = 1, \ldots, n-1$ and $s = h, l$:

$$\frac{\partial}{\partial C^h_j} M(C,X) = (1-\alpha)\beta T \cdot \begin{bmatrix} \left(z^h\right)^2 \frac{B_{j+1}}{\omega^e}\, u''^h & 0 \\ 0 & 0 \end{bmatrix},$$

$$\frac{\partial}{\partial C^l_j} M(C,X) = (1-\alpha)\beta T \cdot \begin{bmatrix} 0 & 0 \\ 0 & \left(z^l\right)^2 \frac{B_{j+1}}{\omega^e}\, u''^l \end{bmatrix}.$$

We infer the derivative with respect to C^s_{n-i} with $i = 1, \ldots, n-1$:

$$\frac{\partial}{\partial C^h_{n-i}} M(C,X) = (1-\alpha)\beta \begin{bmatrix} \pi^h \left(z^h\right)^2 \frac{B_{n-i+1}}{\omega^e} u''^h & 0 \\ \left(1-\pi^l\right) \left(z^h\right)^2 \frac{B_{n-i+1}}{\omega^e} u''^h & 0 \end{bmatrix},$$

$$\frac{\partial}{\partial C^l_{n-i}} M(C,X) = (1-\alpha)\beta \begin{bmatrix} 0 & \left(1-\pi^h\right) \left(z^l\right)^2 \frac{B_{n-i+1}}{\omega^e} u''^l \\ 0 & \pi^l \left(z^l\right)^2 \frac{B_{n-i+1}}{\omega^e} u''^l \end{bmatrix}.$$

For $i = n$ (derivative with respect to C^s_0), we have:

$$\frac{\partial}{\partial C^h_{n-i}} M(C,X) = (1-\alpha)\beta \begin{bmatrix} \pi^h \left(z^h\right)^2 \frac{B_1}{\omega^e} u''^h & 0 \\ \left(1-\pi^l\right) \left(z^h\right)^2 \frac{B_1}{\omega^e} u''^h & 0 \end{bmatrix},$$

$$\frac{\partial}{\partial C^l_{n-i}} M(C,X) = (1-\alpha)\beta \begin{bmatrix} 0 & \left(1-\pi^h\right) \left(z^l\right)^2 \frac{B_1}{\omega^e} u''^l \\ 0 & \pi^l \left(z^l\right)^2 \frac{B_1}{\omega^e} u''^l \end{bmatrix}.$$

d. Derivative of the matrix $M(C,X)$ with respect to C^s_n. We obtain very similar expressions when taking the derivative of $M(C,X)$ with respect to C^s_n:

$$\frac{\partial}{\partial C^h_n} M(C,X) = (1-\alpha)\beta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \frac{\partial}{\partial C^l_n} M(C,X).$$

e. Expression for the derivative of f with respect to C^s_{n-i}, for $i > 0$. We substitute our previous results into the expression for the derivative of f with respect to C^s_{n-i} in (52), for $i > 0$:

$$\frac{\partial f}{\partial C^h_{n-i}} = \begin{bmatrix} 0_{2(i-1)\times 1} \\ -\beta\left(\alpha + (1-\alpha) z^h u'^h\right) \pi^h \\ -\beta\left(\alpha + (1-\alpha) z^h u'^h\right) \left(1-\pi^l\right) \\ 1 \\ 0 \\ 0_{2(n-i)\times 1} \end{bmatrix} \begin{matrix} \\ \leftarrow \text{rank } 2i-1 \\ \\ \leftarrow \text{rank } 2i+1 \\ \\ \\ \end{matrix}$$

$$-\beta(1-\alpha)\left(z^h\right)^2 \frac{B_{n-i+1}}{\omega^e} u''^h \begin{bmatrix} 0_{2\times 2} & \begin{bmatrix} \pi^h & 0 \\ \left(1-\pi^l\right) & 0 \end{bmatrix} & 0_{2\times 2} & \cdots & 0_{2\times 2} \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \begin{bmatrix} \pi^h & 0 \\ \left(1-\pi^l\right) & 0 \end{bmatrix} \\ 0_{2\times 2} & & & \cdots & 0_{2\times 2} \end{bmatrix} \begin{bmatrix} C^h_n \\ C^l_n \\ \vdots \\ C^h_1 \\ C^l_1 \end{bmatrix}.$$

We infer the expression of $\frac{\partial f}{\partial C^h_{n-i}}$, for $i > 0$:

$$\frac{\partial f}{\partial C^h_{n-i}} = \begin{bmatrix} 0_{2(i-1)\times 1} \\ -\beta\left(\alpha + (1-\alpha) z^h u'^h\right)\pi^h \\ -\beta\left(\alpha + (1-\alpha) z^h u'^h\right)(1-\pi^l) \\ 1 \quad {\scriptstyle (Rank\ 2i+1)} \\ 0 \\ 0_{2(n-i)\times 1} \end{bmatrix} - \beta(1-\alpha)\left(z^h\right)^2 \frac{B_{n-i+1}}{\omega^e} u''^h \begin{bmatrix} \pi^h C^h_{n-1} \\ (1-\pi^l) C^h_{n-1} \\ \vdots \\ \pi^h C^h_0 \\ (1-\pi^l) C^h_0 \\ 0_{2\times 1} \end{bmatrix}.$$

Similarly, we obtain the expression for $\frac{\partial f}{\partial C^l_{n-i}}$, for $i > 0$:

$$\frac{\partial f}{\partial C^l_{n-i}} = \begin{bmatrix} 0_{2(i-1)\times 1} \\ -\beta\left(\alpha + (1-\alpha) z^l u'^l\right)(1-\pi^h) \\ -\beta\left(\alpha + (1-\alpha) z^l u'^l\right)\pi^l \\ 0 \\ 1 \quad {\scriptstyle (Rank\ 2i+2)} \\ 0_{2(n-i)\times 1} \end{bmatrix} - \beta(1-\alpha)\left(z^l\right)^2 \frac{B_{n-i+1}}{\omega^e} u''^l \begin{bmatrix} (1-\pi^h) C^l_{n-1} \\ \pi^l C^l_{n-1} \\ \vdots \\ (1-\pi^h) C^l_0 \\ \pi^l C^l_0 \\ 0_{2\times 1} \end{bmatrix}.$$

f. Expression for the derivative of f with respect to C^s_n. The previous expressions simplify and we obtain:

$$\frac{\partial f}{\partial C^h_n} = \begin{bmatrix} 1 \\ 0 \\ 0_{2n\times 1} \end{bmatrix}; \quad \frac{\partial f}{\partial C^l_n} = \begin{bmatrix} 0 \\ 1 \\ 0_{2n\times 1} \end{bmatrix}.$$

g. Conclusion. The solution can be written in a compact form as:

$$\frac{\partial f}{\partial C^s_{n-i}} = \Gamma^s_{n-i} + K^s_{n-i} \text{ for } i = 0, \ldots, n. \tag{53}$$

To simplify expressions, we introduce the following notations:

- Probabilities:

$$\widetilde{\pi}^{hh} \equiv \pi^h,\ \widetilde{\pi}^{lh} \equiv 1-\pi^l,\ \widetilde{\pi}^{ll} \equiv \pi^l,\ \text{and } \widetilde{\pi}^{hl} \equiv 1-\pi^h. \tag{54}$$

- Γ^s_{n-i} is defined by $\Gamma^s_n = [1_{s=h}\ 1_{s=l}\ 0_{2n}]^\top$ and for $i = 1, \ldots, n-1$ by:

$$\Gamma^s_{n-i} = \begin{bmatrix} 0_{2(i-1)\times 1} \\ -\beta\left(\alpha + (1-\alpha) z^s u'^s\right)\widetilde{\pi}^{hs} \\ -\beta\left(\alpha + (1-\alpha) z^s u'^s\right)\widetilde{\pi}^{ls} \\ 1_{s=h} \\ 1_{s=l} \\ 0_{2(n-i)\times 1} \end{bmatrix} \begin{matrix} \\ \\ \\ \longleftarrow\ Rank\ 2i+1 \\ \longleftarrow\ Rank\ 2i+2 \\ \\ \end{matrix}. \tag{55}$$

- we define K^s_{n-i} for $i = 0, \ldots, n$ as:

$$K^s_{n-i} \equiv -\beta\,(1-\alpha)\,(z^s)^2\,\frac{B_{n+1-i}}{\omega^e}\,1_{i>0}\,u''^s \times \left[\widetilde{\pi}^{hs}\,C^s_{n-1}\ \widetilde{\pi}^{ls}\,C^s_{n-1}\ \ldots\ \widetilde{\pi}^{hs}\,C^s_0\ \widetilde{\pi}^{ls}\,C^s_0\ 0\ 0\right]^\top.$$

The Jacobian $Df_Y = (\frac{\partial f}{\partial C^h_n}, \frac{\partial f}{\partial C^l_n}, \ldots, \frac{\partial f}{\partial C^h_{n-i}}, \frac{\partial f}{\partial C^l_{n-i}}, \ldots, \frac{\partial f}{\partial C^h_0}, \frac{\partial f}{\partial C^l_0})$ of f with respect to C is the sum of an upper triangular matrix with only 1s on its diagonal and a matrix that is equal to 0 when $B = 0$ (because $K^s_{n-i} = 0$ if $B = 0$). The Jacobian is thus invertible for $B = 0$. The implicit function theorem allows us to prove that C is a continuous (in fact $\mathcal{C}^1$) function of $\left[B^\top\ Z^\top\right]$ in a neighborhood $\mathcal{V}_1$ of $\left[0_n^\top\ 1_2^\top\right]$. Moreover, we know from above (See *Existence of a no-trade equilibrium without aggregate shocks.*) that if $\left[B^\top\ Z^\top\right] = \left[0_n^\top\ 1_2^\top\right]$, C satisfies the conditions under which the unemployed do not participate in bond markets. By continuity, there exists a neighborhood $\mathcal{V}_2 \subset \mathcal{V}_1$, such that these non-participation conditions are fulfilled if $\left[B^\top\ Z^\top\right] \in \mathcal{V}_2$. □

The lemma establishes that, starting from a no uncertainty/zero net supply situation, a gradual increase in aggregate risk or bond supplies does not cause the yield curve to jump. Since the equilibrium exists in the zero-volume/no aggregate uncertainty case, it also exists when volumes and aggregate risk are sufficiently small (that is, (46) holds).

2. Pricing kernel decomposition Substituting the market-clearing condition $b^e_{t,k} = B_k/\omega^e$ into the Euler equation (44) and rearranging gives the bond-pricing equation and the corresponding pricing kernel components in the proposition.

1.6 Proof of Proposition 5

1a. Impact of bond supplies on the level of the yield curve. We prove the result by induction. Taking the derivative of (51) w.r.t. to B_i, $1 \leq i \leq n$, we get:

$$\frac{\partial C^\zeta_k}{\partial B_i} = \beta \sum_{s=h,l} \widetilde{\pi}^{\zeta s}\left[(\alpha + (1-\alpha)z^s u'^s)\frac{\partial C^s_{k-1}}{\partial B_i} + \frac{(1-\alpha)C^s_{k-1}(z^s)^2 u''^s}{\omega^e}\sum_{j=1}^{n}\left(\frac{\partial C^s_{j-1}}{\partial B_i}B_j + C^s_{i-1}\right)\right], \tag{56}$$

where u'^s and u''^s are defined in (49), and the $\widetilde{\pi}$s in (54).

- The result holds for $k = 1$, since for small bond supplies we have:

$$\frac{\partial C^\zeta_1}{\partial B_i} \approx \beta\frac{(1-\alpha)\,u''\,(\delta)}{\omega^e}\sum_{s=h,l}\widetilde{\pi}^{\zeta s} z^s C^s_{i-1} < 0.$$

- Suppose that the result holds for $k-1$: $\frac{\partial C^h_{k-1}}{\partial B_i}, \frac{\partial C^l_{k-1}}{\partial B_i} < 0$. Since C^s_{j-1} is a $\mathcal{C}^1$ function of B_i, $\frac{\partial C^s_{j-1}}{\partial B_i}$ is continuous in B_i and $B_j\frac{\partial C^s_{j-1}}{\partial B_i}$ is negligible relative to C^s_{i-1} for small bond supplies.

Then, (56) together with the induction assumption ($\frac{\partial C^h_{k-1}}{\partial B_i}, \frac{\partial C^l_{k-1}}{\partial B_i} < 0$) imply that $\frac{\partial C^h_k}{\partial B_i} < 0$, so that a greater bond supply lower bond prices (i.e., raises yields).

1b. Impact of bond supplies on the slope of the yield curve. Diagonalising $M(C,X)$ in (50), we get $M(C,X) = \beta\, Q\, D\, Q^{-1}$, where Q is a 2×2 invertible matrix and $D = Diag(\tilde{\nu}_1, \tilde{\nu}_2)$, with

$$\tilde{\nu}_1 = H + \tilde{\nu}_2 = \frac{1}{2}\left(\alpha\left(\pi^h + \pi^l\right) + (1-\alpha)\left(z^h u'^h \pi^h + z^l u'^l \pi^l\right) + H\right), \text{ and} \tag{57}$$

$$H \equiv \left(\left(\alpha(\pi^h+\pi^l) + (1-\alpha)(z^h\pi^h\, u'^h + z^l\pi^l\, u'^l)\right)^2 - 4(\pi^h+\pi^l-1)(\alpha+(1-\alpha)z^h\, u'^h)(\alpha+(1-\alpha)z^l\, u'^l)\right)^{1/2} > 0.$$

Using Lemma 1, the long yield r^{PV}_∞ is given by:

$$\lim_{k\to\infty} r^h_k = \lim_{k\to\infty} r^l_k = -\ln\beta - \ln\left(\tilde{\nu}_1\right). \tag{58}$$

From (50)–(51) and the fact that $C^s_0 = 1/z^s$, the short yield in state s is:

$$r^s_1 = -\ln p^s_1 = -\ln\beta - \ln\left[\pi^s(\alpha + (1-\alpha)z^s u'^s) + (1-\pi^s)(\alpha z^s/z^{\bar{s}} + (1-\alpha)z^s u'^{\bar{s}})\right],\ s = l, h,$$

where $\bar{s}$ is the state opposite to s.

As in the proof of Proposition 3, we consider a mean preserving spread in z and carry out a second-order Taylor expansion of the derivative of the slope of the yield curve w.r.t. to an increase in the supply of bond of maturity j (i.e., $\frac{\partial\Delta^{PV}}{\partial B_j} = \frac{\partial\left(r^{PV}_\infty - \left((1-\eta^l)r^h_1 + \eta^l r^l_1\right)\right)}{\partial B_j}$) around $\varepsilon = 0$ and zero net volumes. The next section presents a second-order Taylor expansion of Δ^{PV} for the case of i.i.d. shocks. For a general shock process, we only focus on the expansion of $\frac{\partial\Delta^{PV}}{\partial B_j}$. By the definition of Δ^{PV}, we have:

$$\begin{aligned}\left.\frac{\partial\Delta^{PV}}{\partial B_j}\right|_{B=0} &= \left.\frac{\partial r^{PV}_\infty}{\partial B_j}\right|_{B=0} - (1-\eta^l)\left.\frac{\partial r^h_1}{\partial B_j}\right|_{B=0} - \eta^l\left.\frac{\partial r^l_1}{\partial B_j}\right|_{B=0} \\ &= -\frac{1}{\widehat{p}^{ZV}_\infty}\left.\frac{\partial\widehat{p}^{PV}_\infty}{\partial B_j}\right|_{B=0} + (1-\eta^l)\frac{1}{p^{h,ZV}_1}\left.\frac{\partial p^h_1}{\partial B_j}\right|_{B=0} + \eta^l\frac{1}{p^{l,ZV}_1}\left.\frac{\partial p^l_1}{\partial B_j}\right|_{B=0}.\end{aligned} \tag{59}$$

In the above formula, we used the relationship between prices and rates (so that $r^{PV}_\infty = -\ln(\widehat{p}^{PV}_\infty)$, with $\widehat{p}^{PV}_\infty = \beta\tilde{\nu}_1$). We also used the fact that, for all k, $r^{PV}_k\big|_{B=0} = r^{ZV}_k$. We proceed in 3 steps: (i) we take the derivative of $\widehat{p}^{PV}_\infty$ w.r.t. B_j, (ii) we compute the derivative of p^h_1 w.r.t. B_j and (iii) we compute $\left.\frac{\partial\Delta^{PV}}{\partial B_j}\right|_{B=0}$. It is useful to notice the following prelimary result:

$$\left.\frac{\partial u'^s}{\partial B_j}\right|_{B=0} = \frac{p^{ZV,s}_{j-1}}{\omega^e}u''(\delta).$$

We can re-use the results in the preliminary section (Section 1.3) with a constant idiosyncratic shock α (i.e., $a = 0$). We will proceed in two steps. First, we will compute second-order developments of bond prices while keeping the liquidation prices $p^{ZV,s}_{j-1}$ constant. In a second step, we will develop these liquidation prices.

Second-order development of the derivative of $\widehat{p}_\infty^{PV}$ w.r.t. B_j while keeping liquidation prices constant. Using $\widehat{p}_\infty^{PV} = \beta\tilde{\nu}_1$ together with the expression for $\tilde{\nu}_1$ in (57), we infer that the derivative of $\widehat{p}_\infty^{PV}$ w.r.t. B_j can be expressed as follows:

$$\frac{2}{\beta}\left.\frac{\partial \widehat{p}_\infty^{PV}}{\partial B_j}\right|_{B=0} = \frac{(1-\alpha)u''(\delta)}{\omega^e}\left(\pi^h z^h p_{j-1}^{ZV,h} + \pi^l z^l p_{j-1}^{ZV,l} + \frac{N}{D}\right), \tag{60}$$

$$\text{where: } N = \left(\pi^h z^h p_{j-1}^{ZV,h} + \pi^l z^l p_{j-1}^{ZV,l}\right)\left(\pi^h\kappa^h + \pi^l\kappa^l\right) - 2(\pi^h+\pi^l-1)\left(\kappa^l z^h p_{j-1}^{ZV,h} + \kappa^h z^l p_{j-1}^{ZV,l}\right), \tag{61}$$

$$D = \left(\left(\pi^h\kappa^h + \pi^l\kappa^l\right)^2 - 4(\pi^h+\pi^l-1)\kappa^h\kappa^l\right)^{\frac{1}{2}}, \tag{62}$$

$$\text{and: } \theta = \frac{(1-\alpha)u''(\delta)}{\omega^e}, \tag{63}$$

where the $\kappa^{ZV,s}$s are defined in (29) and (30). We also use the following notations:

$$\kappa = \alpha + (1-\alpha)zu'(\delta), \tag{64}$$

$$\Omega = \left((2-\pi^h-\pi^l)\kappa\right)^{-1}. \tag{65}$$

κ as defined in (64) is simply the common value of the $\kappa^{ZV,s}$s when there is no aggregate risk.

Second-order development of the fraction $\frac{N}{D}$ in (60). The numerator N in (60, defined in (61), can be approximated as follows:

$$\begin{aligned}
N &= \left(\pi^h z^h p_{j-1}^{ZV,h} + \pi^l z^l p_{j-1}^{ZV,l}\right)\left(\pi^h\kappa^h + \pi^l\kappa^l\right) - 2(\pi^h+\pi^l-1)\left(\kappa^l z^h p_{j-1}^{ZV,h} + \kappa^h z^l p_{j-1}^{ZV,l}\right)\\
&= \left(\left(2(1-\pi^l) - (2-\pi^h-\pi^l)\pi^h\right)p_{j-1}^{ZV,h} + \left(2(1-\pi^h) - (2-\pi^h-\pi^l)\pi^l\right)p_{j-1}^{ZV,l}\right)\kappa z\\
&+ \frac{2}{2-\pi^h-\pi^l}\left((1-\pi^h)\left(2(1-\pi^l) - (2-\pi^h-\pi^l)\pi^h\right)p_{j-1}^{ZV,h} - (1-\pi^l)\left(2(1-\pi^h) - (2-\pi^h-\pi^l)\pi^l\right)p_{j-1}^{ZV,l}\right)\kappa z\varepsilon\\
&+ 2\frac{\pi^h+\pi^l-1}{2-\pi^h-\pi^l}\left(\left(2(1-\pi^l) - \pi^h(\pi^h-\pi^l)\right)p_{j-1}^{ZV,h} - \left(2(1-\pi^h) + (\pi^h-\pi^l)\pi^l\right)p_{j-1}^{ZV,l}\right)(1-\alpha)z^2u'(\delta)\varepsilon\\
&+ 4\frac{\pi^h+\pi^l-1}{(2-\pi^h-\pi^l)^2}\left((1-\pi^h)\left(2(1-\pi^l) - (\pi^h-\pi^l)\pi^h\right)p_{j-1}^{ZV,h}\right.\\
&\qquad\qquad \left. + (1-\pi^l)\left(2(1-\pi^h) + (\pi^h-\pi^l)\pi^l\right)p_{j-1}^{ZV,l}\right)(1-\alpha)z^2u'(\delta)\varepsilon^2\\
&= \left(\left(2(1-\pi^l) - (2-\pi^h-\pi^l)\pi^h\right)p_{j-1}^{ZV,h} + \left(2(1-\pi^h) - (2-\pi^h-\pi^l)\pi^l\right)p_{j-1}^{ZV,l}\right)\kappa z \qquad (66)\\
&+ \frac{2}{2-\pi^h-\pi^l}\left((1-\pi^h)\left(2(1-\pi^l) - (2-\pi^h-\pi^l)\pi^h\right)p_{j-1}^{ZV,h} - (1-\pi^l)\left(2(1-\pi^h) - (2-\pi^h-\pi^l)\pi^l\right)p_{j-1}^{ZV,l}\right)\alpha z\varepsilon\\
&+ \frac{2}{2-\pi^h-\pi^l}\left(\left(2\pi^l(1-\pi^l) - \pi^h\left((1-\pi^h)(2-\pi^h-\pi^l) + (\pi^h+\pi^l-1)(\pi^h-\pi^l)\right)\right)p_{j-1}^{ZV,h}\right.\\
&\qquad \left. - \left(2\pi^h(1-\pi^h) - \pi^l\left((1-\pi^l)(2-\pi^h-\pi^l) - (\pi^h+\pi^l-1)(\pi^h-\pi^l)\right)\right)p_{j-1}^{ZV,l}\right)(1-\alpha)z^2u'(\delta)\varepsilon\\
&+ 4\frac{\pi^h+\pi^l-1}{(2-\pi^h-\pi^l)^2}\left((1-\pi^h)\left(2(1-\pi^l) - (\pi^h-\pi^l)\pi^h\right)p_{j-1}^{ZV,h}\right.\\
&\qquad\qquad \left. + (1-\pi^l)\left(2(1-\pi^h) + (\pi^h-\pi^l)\pi^l\right)p_{j-1}^{ZV,l}\right)(1-\alpha)z^2u'(\delta)\varepsilon^2.
\end{aligned}$$

Using (35), the denominator D defined in (62) can be approximated as follows:

$$
\begin{aligned}
D^{-1} &= \left(\left(\pi^h \kappa^{ZV,h} + \pi^l \kappa^{ZV,l}\right)^2 - 4(\pi^h + \pi^l - 1)\kappa^{ZV,h}\kappa^{ZV,l}\right)^{-\frac{1}{2}} \\
&= \Omega - 2\frac{(\pi^h + \pi^l - 1)(\pi^h - \pi^l)\Omega^2}{(2 - \pi^h - \pi^l)}(1-\alpha) z u'(\delta)\varepsilon \\
&\quad + 4(\pi^h + \pi^l - 1)\frac{(\pi^h + \pi^l - 1)(\pi^h - \pi^l)^2 - 2(1-\pi^h)(1-\pi^l)}{(2 - \pi^h - \pi^l)^2}\Omega^3 (1-\alpha)^2 u'(\delta)^2 z^2 \varepsilon^2.
\end{aligned}
\tag{67}
$$

Using (66) and (67) provides the following expression for the fraction N/D:

$$
\begin{aligned}
\frac{N}{D} &= \left(\left(2\frac{(1-\pi^l)}{(2-\pi^h-\pi^l)} - \pi^h\right) p_{j-1}^{ZV,h} + \left(2\frac{(1-\pi^h)}{(2-\pi^h-\pi^l)} - \pi^l\right) p_{j-1}^{ZV,l}\right) z \\
&+ \frac{2\Omega}{2-\pi^h-\pi^l}\left((1-\pi^h)\left(2(1-\pi^l) - (2-\pi^h-\pi^l)\pi^h\right) p_{j-1}^{ZV,h} - (1-\pi^l)\left(2(1-\pi^h) - (2-\pi^h-\pi^l)\pi^l\right) p_{j-1}^{ZV,l}\right)\alpha z \varepsilon \\
&+ \frac{2\Omega}{2-\pi^h-\pi^l}\left(\left(2\pi^l(1-\pi^l) - \pi^h\left((1-\pi^h)(2-\pi^h-\pi^l) + (\pi^h+\pi^l-1)(\pi^h-\pi^l)\right)\right) p_{j-1}^{ZV,h}\right. \\
&\qquad \left. - \left(2\pi^h(1-\pi^h) - \pi^l\left((1-\pi^l)(2-\pi^h-\pi^l) - (\pi^h+\pi^l-1)(\pi^h-\pi^l)\right)\right) p_{j-1}^{ZV,l}\right)(1-\alpha) z^2 u'(\delta)\varepsilon \\
&- 2\frac{(\pi^h+\pi^l-1)(\pi^h-\pi^l)\Omega}{(2-\pi^h-\pi^l)}\left(\left(2\frac{(1-\pi^l)}{(2-\pi^h-\pi^l)} - \pi^h\right) p_{j-1}^{ZV,h} + \left(2\frac{(1-\pi^h)}{(2-\pi^h-\pi^l)} - \pi^l\right) p_{j-1}^{ZV,l}\right)(1-\alpha) z^2 u'(\delta)\varepsilon \\
&+ 4\frac{(\pi^h+\pi^l-1)\Omega}{(2-\pi^h-\pi^l)^2}\left((1-\pi^h)\left(2(1-\pi^l) - (\pi^h-\pi^l)\pi^h\right) p_{j-1}^{ZV,h}\right. \\
&\qquad \left. + (1-\pi^l)\left(2(1-\pi^h) + (\pi^h-\pi^l)\pi^l\right) p_{j-1}^{ZV,l}\right)(1-\alpha) z^2 u'(\delta)\varepsilon^2 \\
&+ 4(\pi^h+\pi^l-1)\frac{(\pi^h+\pi^l-1)(\pi^h-\pi^l)^2 - 2(1-\pi^h)(1-\pi^l)}{(2-\pi^h-\pi^l)^2}\Omega^2 \\
&\qquad \times \left(\left(\frac{2(1-\pi^l)}{(2-\pi^h-\pi^l)} - \pi^h\right) p_{j-1}^{ZV,h} + \left(\frac{2(1-\pi^h)}{(2-\pi^h-\pi^l)} - \pi^l\right) p_{j-1}^{ZV,l}\right)(1-\alpha)^2 u'(\delta)^2 z^3 \varepsilon^2 \\
&- \frac{4(\pi^h+\pi^l-1)\Omega^2}{(2-\pi^h-\pi^l)^2}(\pi^h-\pi^l)\left((1-\pi^h)\left(2(1-\pi^l) - (2-\pi^h-\pi^l)\pi^h\right) p_{j-1}^{ZV,h}\right. \\
&\qquad \left. - (1-\pi^l)\left(2(1-\pi^h) - (2-\pi^h-\pi^l)\pi^l\right) p_{j-1}^{ZV,l}\right)\alpha(1-\alpha) u'(\delta) z^2 \varepsilon \\
&- \frac{4(\pi^h+\pi^l-1)\Omega^2}{(2-\pi^h-\pi^l)^2}(\pi^h-\pi^l)\left(\left(2\pi^l(1-\pi^l) - \pi^h\left((1-\pi^h)(2-\pi^h-\pi^l) + (\pi^h+\pi^l-1)(\pi^h-\pi^l)\right)\right) p_{j-1}^{ZV,h}\right. \\
&\qquad \left. - \left(2\pi^h(1-\pi^h) - \pi^l\left((1-\pi^l)(2-\pi^h-\pi^l) - (\pi^h+\pi^l-1)(\pi^h-\pi^l)\right)\right) p_{j-1}^{ZV,l}\right)(1-\alpha)^2 z^3 u'(\delta)^2 \varepsilon^2.
\end{aligned}
\tag{68}
$$

Second-order development of the fraction $\frac{N}{D}$ in (60). At this stage we can simplify equation (68) by noting that the quadratic terms in ε^2, in which $p_{j-1}^{ZV,l}$ and $p_{j-1}^{ZV,h}$ play a role, are already second-order terms. This means that in these expressions, only the zero-order developments of $p_{j-1}^{ZV,l}$ and $p_{j-1}^{ZV,h}$ matter. Moreover, the zero-order term is nothing but the price of a $j-1$ period bond in a world without aggregate risk (i.e., $z^h = z^l$), which is the same in states l and h. We thus denote by p_{j-1}^{ZV} the zero-order approximation of the $j-1$ period bond price. The expression (68) for N/D can thus be simplified as follows:

$$\begin{aligned}
\frac{N}{D} = & \left(\left(\frac{2(1-\pi^l)}{(2-\pi^h-\pi^l)} - \pi^h\right) p_{j-1}^{ZV,h} + \left(\frac{2(1-\pi^h)}{(2-\pi^h-\pi^l)} - \pi^l\right) p_{j-1}^{ZV,l}\right) z \\
& + 2\Omega \left((1-\pi^h)\left(\frac{2(1-\pi^l)}{(2-\pi^h-\pi^l)} - \pi^h\right) p_{j-1}^{ZV,h} - (1-\pi^l)\left(\frac{2(1-\pi^h)}{(2-\pi^h-\pi^l)} - \pi^l\right) p_{j-1}^{ZV,l}\right) \kappa z \varepsilon \\
& + \frac{8(\pi^h+\pi^l-1)(1-\pi^h)(1-\pi^l)}{(2-\pi^h-\pi^l)^2} \Omega \left(p_{j-1}^{ZV,h} - p_{j-1}^{ZV,l}\right)(1-\alpha) z^2 u'(\delta)\varepsilon \\
& + 4\frac{(\pi^h+\pi^l-1)\Omega}{(2-\pi^h-\pi^l)^2} p_{j-1}^{ZV}\left(4(1-\pi^h)(1-\pi^l) + (\pi^h+\pi^l-1)(\pi^h-\pi^l)^2\right)(1-\alpha) z^2 u'(\delta)\varepsilon^2 \\
& + 4\frac{(\pi^h+\pi^l-1)\Omega^2}{(2-\pi^h-\pi^l)} p_{j-1}^{ZV}\left((\pi^h+\pi^l-1)(\pi^h-\pi^l)^2 - 2(1-\pi^h)(1-\pi^l)\right)(1-\alpha)^2 u'(\delta)^2 z^3 \varepsilon^2 \\
& - \frac{4(\pi^h+\pi^l-1)\Omega^2}{(2-\pi^h-\pi^l)} p_{j-1}^{ZV}(\pi^h+\pi^l-1)(\pi^h-\pi^l)^2 \alpha(1-\alpha) u'(\delta) z^2 \varepsilon^2 \\
& - \frac{8(\pi^h+\pi^l-1)\Omega^2}{(2-\pi^h-\pi^l)} p_{j-1}^{ZV}(\pi^h+\pi^l-1)(\pi^h-\pi^l)^2 (1-\alpha)^2 z^3 u'(\delta)^2 \varepsilon^2.
\end{aligned}$$

We obtain, after gathering terms of the same order appropriately:

$$\begin{aligned}
\frac{N}{D} = & \left(\left(\frac{2(1-\pi^l)}{(2-\pi^h-\pi^l)} - \pi^h\right) p_{j-1}^{ZV,h} + \left(\frac{2(1-\pi^h)}{(2-\pi^h-\pi^l)} - \pi^l\right) p_{j-1}^{ZV,l}\right) z \\
& + 2\Omega \left((1-\pi^h)\left(\frac{2(1-\pi^l)}{(2-\pi^h-\pi^l)} - \pi^h\right) p_{j-1}^{ZV,h} - (1-\pi^l)\left(\frac{2(1-\pi^h)}{(2-\pi^h-\pi^l)} - \pi^l\right) p_{j-1}^{ZV,l}\right) \kappa z \varepsilon \\
& + \frac{8(\pi^h+\pi^l-1)(1-\pi^h)(1-\pi^l)}{(2-\pi^h-\pi^l)^2} \Omega \left(p_{j-1}^{ZV,h} - p_{j-1}^{ZV,l}\right)(1-\alpha) z^2 u'(\delta)\varepsilon \\
& + 8\frac{(\pi^h+\pi^l-1)(1-\pi^h)(1-\pi^l)\Omega^2}{(2-\pi^h-\pi^l)} p_{j-1}^{ZV}\left(2\alpha + (1-\alpha) z u'(\delta)\right)(1-\alpha) z^2 u'(\delta)\varepsilon^2.
\end{aligned} \tag{69}$$

Second-order development of the derivative of $\widehat{p}_\infty^{PV}$ (equation (60)). Using the expression for N/D in (69), we infer the derivative $\frac{2}{\beta\theta}\left.\frac{\partial \widehat{p}_\infty^{PV}}{\partial B_j}\right|_{B=0} = \frac{D}{N} + \pi^h z^h p_{j-1}^{ZV,h} + \pi^l z^l p_{j-1}^{ZV,l}$ to be:

$$\begin{aligned}
\frac{1}{\beta\theta}\left.\frac{\partial \widehat{p}_\infty^{PV}}{\partial B_j}\right|_{B=0} = & \frac{(1-\pi^l) p_{j-1}^{ZV,h} + (1-\pi^h) p_{j-1}^{ZV,l}}{(2-\pi^h-\pi^l)} \\
& + \frac{2(1-\pi^h)(1-\pi^l)}{(2-\pi^h-\pi^l)} \Omega \left(p_{j-1}^{ZV,h} - p_{j-1}^{ZV,l}\right) \kappa z \varepsilon \\
& + \frac{4(\pi^h+\pi^l-1)(1-\pi^h)(1-\pi^l)}{(2-\pi^h-\pi^l)^2} \Omega \left(p_{j-1}^{ZV,h} - p_{j-1}^{ZV,l}\right)(1-\alpha) z^2 u'(\delta)\varepsilon \\
& + 4\frac{(\pi^h+\pi^l-1)(1-\pi^h)(1-\pi^l)\Omega^2}{(2-\pi^h-\pi^l)} p_{j-1}^{ZV}\left(2\alpha + (1-\alpha) z u'(\delta)\right)(1-\alpha) z^2 u'(\delta)\varepsilon^2.
\end{aligned} \tag{70}$$

Second-order development of the derivative of r_∞^{PV}. The derivative of the long yield is $\left.\frac{\partial r_\infty^{PV}}{\partial B_j}\right|_{B=0} = -\frac{1}{\widehat{p}_\infty^{ZV}}\left.\frac{\partial \widehat{p}_\infty^{PV}}{\partial B_j}\right|_{B=0}$. In addition to the expression (70) for the derivative of $\widehat{p}_\infty^{ZV}$, we need a second-order approximation of $1/\widehat{p}_\infty^{ZV}$. Using (36), we obtain:

$$\begin{aligned}
\frac{\beta}{\widehat{p}_\infty^{ZV}} = & \left(\alpha + (1-\alpha) z u'(\delta)\right)^{-1} \\
& - 4\Omega^3(\pi^h+\pi^l-1)(1-\pi^h)(1-\pi^l)(1-\alpha)^2 u'(\delta)^2 z^2 \varepsilon^2.
\end{aligned} \tag{71}$$

From (70) and (71), we infer (using the fact that, at the zero order, prices are equal to p_{j-1}^{ZV}):

$$\begin{aligned} \frac{1}{z\theta}\frac{1}{\widehat{p}_\infty^{ZV}}\left.\frac{\partial \widehat{p}_\infty^{PV}}{\partial B_j}\right|_{B=0} &= \Omega\left((1-\pi^l)p_{j-1}^{ZV,h} + (1-\pi^h)p_{j-1}^{ZV,l}\right) \\ &+ 2(1-\pi^h)(1-\pi^l)\Omega^2\left(p_{j-1}^{ZV,h} - p_{j-1}^{ZV,l}\right)\kappa\varepsilon \\ &+ \frac{4(\pi^h+\pi^l-1)(1-\pi^h)(1-\pi^l)}{(2-\pi^h-\pi^l)}\Omega^2\left(p_{j-1}^{ZV,h} - p_{j-1}^{ZV,l}\right)(1-\alpha)zu'(\delta)\varepsilon \\ &+ 8(\pi^h+\pi^l-1)(1-\pi^h)(1-\pi^l)\Omega^3 p_{j-1}^{ZV}\alpha(1-\alpha)zu'(\delta)\varepsilon^2. \end{aligned} \tag{72}$$

Second-order development of the derivative of p_1^h w.r.t. B_j. We now develop $\frac{1}{p_1^{h,ZV}}\left.\frac{\partial p_1^h}{\partial B_j}\right|_{B=0}$. Using the fact that $p_1^s = C_1 z^s$, we deduce that $\frac{1}{p_1^{h,ZV}}\left.\frac{\partial p_1^h}{\partial B_j}\right|_{B=0} = \frac{1}{C_1^{h,ZV}}\left.\frac{\partial C_1^h}{\partial B_j}\right|_{B=0}$. From (51), we have: $C_1^h = \beta\pi^h(\alpha+(1-\alpha)\,z^h\,u'^h)\frac{1}{z^h} + \beta(1-\pi^h)(\alpha+(1-\alpha)\,z^l\,u'^l)\frac{1}{z^l}$ and, computing the derivative:

$$\left.\frac{\partial C_1^h}{\partial B_j}\right|_{B=0} = \beta(1-\alpha)\frac{1}{\omega^e}u''(\delta)\left(\pi^h p_{j-1}^{ZV,h} + (1-\pi^h)p_{j-1}^{ZV,l}\right). \tag{73}$$

We now use (21), which provides a second-order development of $C_1^{ZV,h}$. We obtain:

$$\begin{aligned} \frac{\beta}{zC_1^{ZV,h}} &= \left(\alpha+(1-\alpha)zu'(\delta)\right)^{-1} + 2\alpha(1-\pi^h)(2-\pi^h-\pi^l)\left(\pi^h+\pi^l-1\right)\Omega^2\varepsilon \\ &+ 4\alpha(1-\pi^h)\Omega^2\left(\alpha\left(1-\pi^h\right)(2-\pi^h-\pi^l)\left(\pi^h+\pi^l-1\right)^2\Omega - (1-\pi^h)(\pi^h+\pi^l-1) - (1-\pi^l)(2-\pi^h-\pi^l)\right) \\ &= \left(\alpha+(1-\alpha)zu'(\delta)\right)^{-1} + 2\alpha(1-\pi^h)(2-\pi^h-\pi^l)\left(\pi^h+\pi^l-1\right)\Omega^2\varepsilon \\ &+ 4\alpha(1-\pi^h)\Omega^3(2-\pi^h-\pi^l)\left(\alpha\pi^h(2-\pi^h-\pi^l)^2 - \left(\pi^h(1-\pi^h)+(1-\pi^l)^2\right)(1-\alpha)zu'(\delta)\varepsilon^2\right). \end{aligned} \tag{74}$$

From (73) and (74), we deduce the following expression for $\frac{1}{p_1^{h,ZV}}\left.\frac{\partial p_1^h}{\partial B_j}\right|_{B=0} = \frac{1}{C_1^{h,ZV}}\left.\frac{\partial C_1^h}{\partial B_j}\right|_{B=0}$:

$$\begin{aligned} \frac{1}{z\theta p_1^{h,ZV}}\left.\frac{\partial p_1^h}{\partial B_j}\right|_{B=0} &= \left(\pi^h p_{j-1}^{ZV,h} + (1-\pi^h)p_{j-1}^{ZV,l}\right) \\ &\times\left(\left(\alpha+(1-\alpha)zu'(\delta)\right)^{-1} + 2\alpha(1-\pi^h)(2-\pi^h-\pi^l)\left(\pi^h+\pi^l-1\right)\Omega^2\varepsilon\right. \\ &\left.-4\alpha(1-\pi^h)\Omega^3(2-\pi^h-\pi^l)\left(\alpha\pi^h(2-\pi^h-\pi^l)^2 + \left(\pi^h(1-\pi^h)+(1-\pi^l)^2\right)(1-\alpha)zu'(\delta)\right)\varepsilon^2\right). \end{aligned} \tag{75}$$

Similarly, we obtain, for the yield curve in the state l:

$$\begin{aligned} \frac{1}{z\theta p_1^{l,ZV}}\left.\frac{\partial p_1^l}{\partial B_j}\right|_{B=0} &= \left(\pi^l p_{j-1}^{ZV,l} + (1-\pi^l)p_{j-1}^{ZV,h}\right) \\ &\times\left(\left(\alpha+(1-\alpha)zu'(\delta)\right)^{-1} - 2\alpha(1-\pi^l)(2-\pi^h-\pi^l)\left(\pi^h+\pi^l-1\right)\Omega^2\varepsilon\right. \\ &\left.-4\alpha(1-\pi^l)\Omega^3(2-\pi^h-\pi^l)\left(\alpha\pi^l(2-\pi^h-\pi^l)^2 + \left(\pi^l(1-\pi^l)+(1-\pi^h)^2\right)(1-\alpha)zu'(\delta)\right)\varepsilon^2\right). \end{aligned} \tag{76}$$

Substituting (72), (75), and (76) into (59), we find:

$$
\begin{aligned}
\frac{1}{z\theta}\left.\frac{\partial\Delta^{PV}}{\partial B_j}\right|_{B=0} &= \left(\alpha+(1-\alpha)zu'(\delta)\right)^{-1}\left(\frac{(1-\pi^l)}{2-\pi^h-\pi^l}p_{j-1}^{ZV,h}+\frac{(1-\pi^h)}{2-\pi^h-\pi^l}p_{j-1}^{ZV,l}\right)\\
&-\Omega\left((1-\pi^l)p_{j-1}^{ZV,h}+(1-\pi^h)p_{j-1}^{ZV,l}\right)\\
&+2\alpha(1-\pi^h)(1-\pi^l)\left(\pi^h+\pi^l-1\right)^2\Omega^2\left(p_{j-1}^{ZV,h}-p_{j-1}^{ZV,l}\right)\varepsilon\\
&-2(1-\pi^h)(1-\pi^l)\Omega^2\left(p_{j-1}^{ZV,h}-p_{j-1}^{ZV,l}\right)\kappa\varepsilon\\
&-\frac{4(\pi^h+\pi^l-1)(1-\pi^h)(1-\pi^l)}{(2-\pi^h-\pi^l)}\Omega^2\left(p_{j-1}^{ZV,h}-p_{j-1}^{ZV,l}\right)(1-\alpha)zu'(\delta)\varepsilon\\
&-4\alpha(1-\pi^h)(1-\pi^l)\Omega^3p_{j-1}^{ZV}(2-\pi^h-\pi^l)\left(\alpha(\pi^h+\pi^l)(2-\pi^h-\pi^l)+(1-\alpha)zu'(\delta)\right)\varepsilon^2\\
&-4\alpha(1-\pi^h)(1-\pi^l)\Omega^3p_{j-1}^{ZV}2(\pi^h+\pi^l-1)(1-\alpha)zu'(\delta)\varepsilon^2.
\end{aligned}
$$

Gathering terms appropriately, we obtain:

$$
\begin{aligned}
\frac{1}{z\theta}\left.\frac{\partial\Delta^{PV}}{\partial B_j}\right|_{B=0} &= -2(1-\pi^h)(1-\pi^l)(\pi^h+\pi^l)\Omega^3\\
&\times\left(\kappa\left(p_{j-1}^{ZV,h}-p_{j-1}^{ZV,l}\right)+2\alpha p_{j-1}^{ZV}\varepsilon\right)\left(\alpha(2-\pi^h-\pi^l)^2+(1-\alpha)zu'(\delta)\right)\varepsilon.
\end{aligned}
\tag{77}
$$

From (72) and (76), we infer the expression for the derivative of the conditional slope $\Delta^{PV,l} = r_\infty^{PV} - r_1^{PV,l}$ in state l w.r.t B_j:

$$
\begin{aligned}
\frac{1}{z\theta}\left.\frac{\partial\Delta^{PV,l}}{\partial B_j}\right|_{B=0} &= \Omega\left((2-\pi^h-\pi^l)(\pi^l p_{j-1}^{ZV,l}+(1-\pi^l)p_{j-1}^{ZV,h})-(1-\pi^l)p_{j-1}^{ZV,h}-(1-\pi^h)p_{j-1}^{ZV,l}\right)\\
&-2p_{j-1}^{ZV}\alpha(1-\pi^l)(2-\pi^h-\pi^l)\left(\pi^h+\pi^l-1\right)\Omega^2\varepsilon\\
&= -\Omega(1-\pi^l)(\pi^h+\pi^l-1)\left((p_{j-1}^{ZV,h}-p_{j-1}^{ZV,h})+2p_{j-1}^{ZV}\alpha\kappa^{-1}\varepsilon\right).
\end{aligned}
\tag{78}
$$

Second-order development of the derivative of the slope Δ^{PV} w.r.t. B_j . In order to obtain the second-order development of the derivative of the slope, we need to compute the Taylor expansions of $p_{j-1}^{ZV,h} - p_{j-1}^{ZV,l}$ and p_{j-1}^{ZV}, which appear in (77).

Taylor expansions of $p_{j-1}^{ZV,h} - p_{j-1}^{ZV,l}$ and p_{j-1}^{ZV}. We show by induction that:

$$
p_{j-1}^{ZV} = \beta^{j-1}\kappa^{j-1}, \tag{79}
$$

$$
p_{j-1}^{ZV,h} - p_{j-1}^{ZV,l} = 1_{j>1}2\beta^{j-1}\sum_{k=0}^{j-2}(\pi^h+\pi^l-1)^k\kappa^{j-2}\left(\alpha(2-\pi^h-\pi^l)+(1-\alpha)zu'(\delta)\right)\varepsilon. \tag{80}
$$

Proof of equalities (79) and (80).

From (51), it is straightforward that, at the zero order, $p_j^{ZV} = \beta\left(\alpha+(1-\alpha)zu'(\delta)\right)p_{j-1}^{ZV}$, which together with $p_0^{ZV} = 1$ gives (79) .

At the first-order, noticing that $\frac{z^h}{z^l} = 2\varepsilon$, we infer from (51) that the price $p_j^{ZV,h}$ can be expressed as follows:

$$\frac{p_j^{ZV,h}}{\beta} = \pi^h \left(\alpha + (1-\alpha) z^h u'(\delta)\right) p_{j-1}^{ZV,h} + (1-\pi^h)\left(\alpha \frac{z^h}{z^l} + (1-\alpha) z^h u'(\delta)\right) p_{j-1}^{ZV,l}$$

$$= \left(\alpha + (1-\alpha) z u'(\delta)\right)\left(\pi^h p_{j-1}^{ZV,h} + (1-\pi^h) p_{j-1}^{ZV,l}\right) \tag{81}$$

$$+ 2\left((1-\pi^h)\alpha p_{j-1}^{ZV,l} + \frac{1-\pi^h}{2-\pi^h-\pi^l}(1-\alpha) z u'(\delta)\left(\pi^h p_{j-1}^{ZV,h} + (1-\pi^h) p_{j-1}^{ZV,l}\right)\right)\varepsilon$$

$$= \kappa\left(\pi^h p_{j-1}^{ZV,h} + (1-\pi^h) p_{j-1}^{ZV,l}\right) + 2 p_{j-1}^{ZV}\left((1-\pi^h)\alpha + \frac{1-\pi^h}{2-\pi^h-\pi^l}(1-\alpha) z u'(\delta)\right)\varepsilon, \tag{82}$$

where the last equality stems from the fact that we only care about first order and that at the order zero $p_{j-1}^{ZV,h} = p_{j-1}^{ZV,l} = p_{j-1}^{ZV}$. By the same token, we have for the price in state l:

$$\frac{p_j^{ZV,l}}{\beta} = \kappa\left(\pi^l p_{j-1}^{ZV,l} + (1-\pi^l) p_{j-1}^{ZV,h}\right) - 2 p_{j-1}^{ZV}\left((1-\pi^l)\alpha + \frac{1-\pi^l}{2-\pi^h-\pi^l}(1-\alpha) z u'(\delta)\right)\varepsilon. \tag{83}$$

Using (82) and (83), we have:

$$\frac{p_j^{ZV,h} - p_j^{ZV,l}}{\beta} = \kappa(\pi^h + \pi^l - 1)\left(p_{j-1}^{ZV,h} - p_{j-1}^{ZV,l}\right) + 2 p_{j-1}^{ZV}\left((2-\pi^h-\pi^l)\alpha + (1-\alpha) z u'(\delta)\right)\varepsilon. \tag{84}$$

First, for $j = 1$, we have $\frac{p_1^{ZV,h} - p_1^{ZV,l}}{\beta} = 2\left((2-\pi^h-\pi^l)\alpha + (1-\alpha) z u'(\delta)\right)\varepsilon$, which is (80) at the first step.

Using (80) and the expression for p_{j-1}^{ZV}, we obtain (84):

$$\frac{p_j^{ZV,h} - p_j^{ZV,l}}{\beta} = 2\beta^{j-1}\sum_{k=0}^{j-2}(\pi^h + \pi^l - 1)^{k+1}\kappa^{j-1}\left(\alpha(2-\pi^h-\pi^l) + (1-\alpha) z u'(\delta)\right)\varepsilon$$

$$+ 2\beta^{j-1}\kappa^{j-1}\left((2-\pi^h-\pi^l)\alpha + (1-\alpha) z u'(\delta)\right)\varepsilon$$

$$= 2\beta^{j-1}\sum_{k=0}^{j-1}(\pi^h + \pi^l - 1)^{k}\kappa^{j-1}\left(\alpha(2-\pi^h-\pi^l) + (1-\alpha) z u'(\delta)\right)\varepsilon,$$

which concludes the proof of (80). □

Derivative of the slope of the yield curve. Substituting (79) and (80) into (77), we infer that

$$\frac{1}{z\theta}\left.\frac{\partial \Delta^{PV}}{\partial B_j}\right|_{B=0} = -4(1-\pi^h)(1-\pi^l)(\pi^h+\pi^l)\Omega^3 p_{j-1}^{ZV}\left(\alpha(2-\pi^h-\pi^l)^2 + (1-\alpha) z u'(\delta)\right) \tag{85}$$

$$\left(\alpha + 1_{j>1}\sum_{k=0}^{j-2}(\pi^h+\pi^l-1)^k\left(\alpha(2-\pi^h-\pi^l) + (1-\alpha) z u'(\delta)\right)\right)\varepsilon^2,$$

Since $\theta < 0$ (by equation (63)), this completes the proof of the impact of bond volumes on the slope of the yield curve.

Going through the same steps, we find, for the conditional slopes:

$$\frac{1}{z\theta}\left.\frac{\partial\Delta^{PV,l}}{\partial B_j}\right|_{B=0} = -2\Omega(1-\pi^l)(\pi^h+\pi^l-1)\beta^{j-1}\kappa^{j-2} \times\left(\alpha+1_{j>1}\sum_{k=0}^{j-2}(\pi^h+\pi^l-1)^k\left(\alpha(2-\pi^h-\pi^l)+(1-\alpha)zu'(\delta)\right)\right)\varepsilon. \tag{86}$$

4. Explicit formulas in the i.i.d. case First, we have $C_0^h=(1+\varepsilon)^{-1}$ and $C_0^l=(1-\varepsilon)^{-1}$. Using the recursion (50)–(51) to compute C_1^s, $s=l,h$ and rearranging, we find

$$C_1^s=\frac{\alpha\beta}{1-\varepsilon^2}+(1-\alpha)\beta u'(\delta)+\frac{(1-\alpha)\beta u''(\delta)}{2}\left[(1+\varepsilon)\sum_{j=1}^n C_{j-1}^h\frac{B_j}{\omega^e}+(1-\varepsilon)\sum_{j=1}^n C_{j-1}^l\frac{B_j}{\omega^e})\right],$$

which in turn implies that $C_1^h=C_1^l\equiv C_1$. The same recursion gives, for $j\geq 2$,

$$\frac{C_j^s}{C_{j-1}^s}=\alpha\beta+\beta(1-\alpha)u'(\delta)+\beta\frac{(1-\alpha)u''(\delta)}{2}\left[(1+\varepsilon)\sum_{j=1}^n C_{j-1}^h\frac{B_j}{\omega^e}+(1-\varepsilon)\sum_{j=1}^n C_{j-1}^l\frac{B_j}{\omega^e})\right].$$

By induction, $C_j^h=C_j^l\equiv C_j$ for all $j\geq 1$, so the latter two equations can be written as:

$$C_1=\frac{\alpha\beta}{1-\varepsilon^2}+(1-\alpha)\beta u'(\delta)+(1-\alpha)\beta u''(\delta)\left(\frac{B_1}{\omega^e}+\sum_{j=2}^n C_{j-1}\frac{B_j}{\omega^e}\right), \tag{87}$$

$$\frac{C_j}{C_{j-1}}=\beta\alpha+\beta(1-\alpha)u'(\delta)+\beta(1-\alpha)u''(\delta)\left(\frac{B_1}{\omega^e}+(1+\varepsilon^2)\sum_{j=2}^n C_{j-1}\frac{B_j}{\omega^e}\right). \tag{88}$$

Equations (87)–(88) define a system of n equations with n unknown, the C_js. The solution to this system expresses the vector $[C_j]_{j=1}^n$ as a function of ε^2, and for small shocks we have $C_j\simeq\overline{C}_j+\left(\partial_\varepsilon^2 C_j\right)\varepsilon^2$, $j=1,...n$, where $\overline{C}_j$ is the value of C_j without aggregate shocks and $\left(\partial_\varepsilon^2 C_j\right)\equiv\partial C_j/\partial\varepsilon^2\big|_{\varepsilon^2=0}$ (both the $\overline{C}_j$s and the $\left(\partial_\varepsilon^2 C_j\right)$s are undetermined coefficients at this stage). Moreover, we define $\overline{W}\equiv\frac{1}{\omega^e}\sum_{j=1}^n\bar{p}_{j-1}B_j=\frac{1}{\omega^e}\sum_{j=1}^n\overline{C}_{j-1}B_j$ as the value of the portfolio without aggregate shocks, $\overline{W}_2\equiv\overline{W}-B_1/\omega^e=\frac{1}{\omega^e}\sum_{j=2}^n\overline{C}_{j-1}B_j$ the same value excluding holdings of one-period bonds, and $\left(\partial_\varepsilon^2 W\right)\equiv\frac{1}{\omega^e}\sum_{j=2}^n\left(\partial_\varepsilon^2 C_{j-1}\right)B_j$ as the change in the value of the portfolio following a marginal change in ε^2. Computing the first-order approximations to the right hand sides of (87)–(88) around $\varepsilon^2=0$, we get

$$C_1\simeq\underbrace{\alpha\beta+(1-\alpha)\beta u'(\delta)+(1-\alpha)\beta u''(\delta)\overline{W}}_{=\bar{C}_1}+\underbrace{\left(\alpha\beta+(1-\alpha)\beta u''(\delta)\left(\partial_\varepsilon^2 W\right)\right)}_{=\left(\partial_\varepsilon^2 C_1\right)}\varepsilon^2, \tag{89}$$

$$\frac{C_j}{C_{j-1}}=\underbrace{\alpha\beta+(1-\alpha)\beta u'(\delta)+(1-\alpha)\beta u''(\delta)\overline{W}}_{=\bar{C}_1}+\underbrace{\beta(1-\alpha)u''(\delta)\left(\overline{W}_2+\left(\partial_\varepsilon^2 W\right)\right)}_{\equiv\mu}\varepsilon^2. \tag{90}$$

From (90), we have, for $j\geq 2$, $C_j=C_{j-1}\bar{C}_1+C_{j-1}\mu\varepsilon^2$. Using this recursion starting at $C_1=\bar{C}_1+\left(\partial_\varepsilon^2 C_1\right)\varepsilon^2$ and neglecting terms in ε^4, we find that, for $j\geq 1$,

$$C_j\simeq\left(\overline{C}_1\right)^j+\left(\overline{C}_1\right)^{j-1}\left(\left(\partial_\varepsilon^2 C_1\right)+(j-1)\mu\right)\varepsilon^2,$$

where $\bar{C}_j=\left(\overline{C}_1\right)^j$. Now substitute the values for $\left(\partial_\varepsilon^2 C_1\right)$ and μ in (89) and (90) into the latter expression to find

$$C_j \simeq \left(\overline{C}_1\right)^j + \left(\overline{C}_1\right)^{j-1} (\alpha\beta + (1-\alpha)\beta u''(\delta)\left[(j-1)\overline{W}_2 + j\left(\partial_\varepsilon^2 W\right)\right])\varepsilon^2.$$

For small bond volumes, the terms in $\bar{W}$, $\overline{W}_2$ and $\left(\partial_\varepsilon^2 W\right)$ (which include the B_js) are second-order relative to $\alpha\beta$, so the latter equation gives $C_j \simeq \left(\overline{C}_1\right)^j + \alpha\beta\left(\overline{C}_1\right)^{j-1}\varepsilon^2$, where $\overline{C}_1 \simeq \alpha\beta + (1-\alpha)\beta u'(\delta)$ (by (89)). Since $C_j \simeq \overline{C}_j + \left(\partial_\varepsilon^2 C_j\right)\varepsilon^2$, this implies that $\left(\partial_\varepsilon^2 C_j\right) \simeq \alpha\beta\left(\overline{C}_1\right)^{j-1}$, which in turn gives $C_j \simeq \left(\overline{C}_1\right)^j + \alpha\beta\left(\overline{C}_1\right)^{j-1}\varepsilon^2$. We infer $\left(\partial_\varepsilon^2 W\right)$ to be:

$$\left(\partial_\varepsilon^2 W\right) = \sum_{j=2}^{n} \frac{\left(\partial_\varepsilon C_{j-1}\right) B_j}{\omega^e} \simeq \sum_{j=2}^{n} \frac{\alpha\beta\left(\overline{C}_1\right)^{j-2} B_j}{\omega^e} = \frac{\alpha\beta\sum_{j=2}^{n}\left(\overline{C}_1\right)^{j-1} B_j}{\omega^e\overline{C}_1} = \frac{\alpha\overline{W}_2}{\alpha + (1-\alpha)u'(\delta)}. \tag{91}$$

From (58), the long yield in the i.i.d. case is

$$r_\infty^{PV} = -\ln(\beta) - \ln\left(\alpha + \frac{1-\alpha}{2}((1+\varepsilon)u'^h + (1-\varepsilon)u'^l)\right), \tag{92}$$

with $u'^s = u'\left(\delta + \frac{1}{\omega^e}\sum_{j=1}^{n} p_{j-1}^s B_j\right)$. Since $p_{j-1}^s = C_{j-1}z^s$ for $j \geq 2$ and $p_0^s = 1$, we have

$$u'^s = u'\left(\delta + \frac{B_1}{\omega^e} + \frac{1 \pm \varepsilon}{\omega^e}\sum_{j=2}^{n} C_{j-1}B_j\right) \simeq u'(\delta) + u''(\delta)\left(\frac{B_1}{\omega^e} + \frac{1 \pm \varepsilon}{\omega^e}\sum_{j=2}^{n} C_{j-1}B_j\right),$$

and hence, again neglecting terms in ε^4,

$$\begin{aligned}
\frac{(1+\varepsilon)u'^h + (1-\varepsilon)u'^l)}{2} &= u'(\delta) + u''(\delta)\left(\frac{B_1}{\omega^e} + \frac{1+\varepsilon^2}{\omega^e}\sum_{j=2}^{n} C_{j-1}B_j\right) \\
&\simeq u'(\delta) + u''(\delta)\left(\bar{W} + \frac{1}{\omega^e}.\frac{\partial\left(1+\varepsilon^2\right)\sum_{j=2}^{n} C_{j-1}B_j}{\partial\varepsilon^2}.\varepsilon^2\right) \\
&= u'(\delta) + u''(\delta)\left(\bar{W} + \left(\bar{W}_2 + \left(\partial_\varepsilon^2 W\right)\right)\varepsilon^2\right).
\end{aligned}$$

Substituting this expression into (92) and using the value of $\left(\partial_\varepsilon^2 W\right)$ in (91), we find

$$r_\infty^{PV} = -\ln(\beta) - \ln\left(\alpha + (1-\alpha)u'(\delta) + (1-\alpha)u''(\delta)\left(\bar{W} + \bar{W}_2\varepsilon^2 + \frac{\alpha\overline{W}_2}{\alpha + (1-\alpha)u'(\delta)}\varepsilon^2\right)\right) \tag{93}$$

The linearisation of (93) around $\left(\bar{W}, \bar{W}_2\right) = (0,0)$, with $\bar{W}_2 = \bar{W} - B_1/\omega^e = \bar{W} - b_1$, gives **(26)** in the body of the paper.

Let us now turn to the short yield. Under i.i.d. shocks, the average short yield is:

$$\bar{r}_1^{PV} = -\frac{1}{2}\sum_{s=l,h}\ln C_1 z^s = -\ln C_1 - \frac{\ln\left(1-\varepsilon^2\right)}{2}$$

With ε^2 small, we have $-\ln\left(1-\varepsilon^2\right)/2 \simeq \varepsilon^2/2$, while C_1 is given by (89) and $\left(\partial_\varepsilon^2 W\right)$ by (91)). This gives:

$$\bar{r}_1^{PV} \simeq \frac{\varepsilon^2}{2} - \ln\beta - \ln\left(\alpha + (1-\alpha)u'(\delta) + (1-\alpha)u''(\delta)\overline{W} + \alpha\varepsilon^2 + \frac{\alpha(1-\alpha)u''(\delta)\overline{W}_2}{\alpha + (1-\alpha)u'(\delta)}\varepsilon^2\right)$$

Linearising the latter expression around $\left(\bar{W}, \bar{W}_2\right) = 0^2$, we obtain:

$$\begin{aligned} r_1^{PV} \simeq & -\ln(\beta) + \frac{\varepsilon^2}{2} - \ln\left(\alpha + (1-\alpha)u'(\delta) + \alpha\varepsilon^2\right) \\ & - \frac{(1-\alpha)u''(\delta)\overline{W}}{\alpha + (1-\alpha)u'(\delta) + \alpha\varepsilon^2} - \frac{\alpha(1-\alpha)u''(\delta)\varepsilon^2}{\left[\alpha + (1-\alpha)u'(\delta) + \alpha\varepsilon^2\right]\left[\alpha + (1-\alpha)u'(\delta]\right)}\overline{W}_2. \end{aligned} \tag{94}$$

For small ε^2 small, this expression gives **(27)** in the body of the paper.

2 Relaxed model

In this section, we examine the robustness of our theoretical results by relaxing some of the assumptions under which they were derived. To be more specific, we now consider an economy in which i) the aggregate state has continuous rather than discrete support, ii) agents do not instantaneously liquidate their asset wealth, iii) agents may trade a positive-supply asset whose payoff is contingent on the aggregate state, iv) the tax structure is more general than in the baseline model, and v) the supply of bonds is time-varying and indexed on the aggregate state. Allowing for asset liquidation to be gradual rather than immediate implies that we must resort to numerical methods to solve the model. However, the structure of the equilibrium remains sufficiently simple so that we can solve the model via perturbation methods and thereby consider a large number of bond maturities.

As in the baseline model, agents face idiosyncratic unemployed risk and can self-insure against these shocks by purchasing bonds of various maturities. In addition, there is a risky asset (a Lucas tree) that pays out a stochastic dividend y_t in period t. Any agent $i \in [0,1]$ buys an amount x_t^i of a risky asset at a price Q_t in period t. Agents cannot short-sell the asset and hence face the constraint $x_t^i \geq 0$. Moreover, agent i now pays taxes conditional on his or her employment status e_t^i. The tax paid in period t is denoted $\tau_t(e_t^i)$. The program of an agent i can be expressed as follows:

$$\max E_0^i \sum_{t=0}^{\infty} \beta^t \left(u\left(c_t^i\right) - l_t^i\right) \tag{95}$$

$$\text{s.t.} \quad c_t^i + \tau_t(e_t^i) + Q_t x_t^i + \sum_{k=1}^{n} p_{t,k}\, b_{t,k}^i = \sum_{k=1}^{n} p_{t,k-1} b_{t-1,k}^i$$

$$+x_{t-1}^i \left(Q_t + y_t\right) + e_t^i z_t l_t^i + \left(1 - e_t^i\right)\delta, \tag{96}$$

$$Q_t x_t^i + \sum_{k=1}^{n} p_{t,k}\, b_{t,k}^i \geq 0 \tag{97}$$

$$c_t^i,\ l_t^i \geq 0, \tag{98}$$

$$\lim_{t\to\infty} \beta^t u'\left(c_t\right) b_{t,k}^i = 0, \qquad \text{for } k = 1, \ldots, n. \tag{99}$$

$$\lim_{t\to\infty} \beta^t u'\left(c_t\right) x_t^i = 0. \tag{100}$$

We now specify the process for the aggregate state. The productivity level (equal to the real wage in equilibrium) now has the following structure:

$$z_t = \exp\left(v_t\right) \quad \text{with} \quad v_t = \lambda_v v_{t-1} + \sigma_v \varepsilon_t, \tag{101}$$

where $\varepsilon_t \sim \mathcal{N}\left(0,1\right)$ is a white noise following a standard Gaussian distribution and λ_v is the auto-correlation of v_t. We define the steady state of the model as the case where $\sigma_v = 0$. We assume that the aggregate state affects not only the productivity of employed agents but also job transition rates and the tree's dividend. The dependence of all these variables on the aggregate state is meant to capture in a simple fashion the strong co-movements between macroeconomic variables that are observed over the business cycle. More specifically, the transition rates evolve as follows:

$$\alpha_t = \bar{\alpha} + \sigma_\alpha \left(e^{v_t} - 1\right), \tag{102}$$

$$\rho_t = \bar{\rho} + \sigma_\rho \left(e^{v_t} - 1\right). \tag{103}$$

The steady state values of these probabilities are denoted $\bar{\alpha}$ and $\bar{\rho}$. Finally, the process for the dividend y_t of the risky asset is procyclical and is directly related to the technology shock:

$$y_t = \sigma_y e^{v_t}. \tag{104}$$

The government can issue time-varying volumes of bonds of any maturity, $A_{t,k}$. Bond volumes are assumed to be countercyclical, i.e., they fall in good times (high z_t) and rise in bad times (low

z_t). For simplicity, we posit the following linear relationship:

$$A_{t,k} = A_k - \mu_k z_t, \quad k = 1, \ldots, n. \tag{105}$$

The coefficient μ_k scales the size of these movements for each maturity k. This pattern of public debt is akin to a countercyclical fiscal policy, since it implies that, as the economy switches from a boom to a recession, taxes rise less (or fall more) than with constant bond issues. The total supply of bonds of maturity k in period t is now

$$B_{t,k} \equiv \sum_{j=0}^{n-k} A_{t-j,k+j}, \; k = 1, \ldots, n. \tag{106}$$

In the relax model, we allow the tax system to depend on an agent's idiosyncratic state. The lump sum tax on unemployed and employed agents are denoted τ_t^u and τ_t^e, respectively. and the one on employed agents is in period t. The government budget constraint is

$$\omega_t^e \tau_t^e + \omega_t^u \tau_t^u + \sum_{j=1}^{n} p_{t,j} A_{t,j} = \sum_{j=1}^{n} A_{t-j,j}, \tag{107}$$

where ω_t^e and ω_t^u are the time-varying number of employed and unemployed agents consistent with the transition rates α_t and ρ_t.

To determine the evolution of these quantities, we define $\omega_t^{kj}(k, j = e, u)$ as the share of agents in idiosyncratic state j in period t after having been in state k in period $t-1$. The laws of motion for these shares are:

$$\omega_t^{ee} = (1-\alpha_t)\left(\omega_{t-1}^{ee} + \omega_{t-1}^{ue}\right), \qquad \omega_t^{uu} = \rho_t\left(\omega_{t-1}^{eu} + \omega_{t-1}^{uu}\right), \tag{108}$$

$$\omega_t^{eu} = \alpha_t\left(\omega_{t-1}^{ee} + \omega_{t-1}^{ue}\right), \qquad \omega_t^{ue} = (1-\rho_t)\left(\omega_{t-1}^{eu} + \omega_{t-1}^{uu}\right). \tag{109}$$

We assume that these shares are at their steady state values at date 0. ω_t^e and ω_t^u are given by:

$$\omega_t^u = \omega_t^{eu} + \omega_t^{uu} \quad \text{and} \quad \omega_t^e = \omega_t^{ue} + \omega_t^{ee}. \tag{110}$$

Market equilibria. Defining a time-varying measure over the set of agents, the total demand for the risky asset must be equal to V in each period and the total demand for bonds of maturity k must be equal to $B_{t,k}$ in period t:

$$\int_{(b_1,\ldots,b_k,\ldots,b_n,x,e)\in(\mathbb{R}^+)^{n+1}\times E} b_{k,t}\, d\Lambda_t\left(b_1,\ldots,b_k,\ldots,b_n,x,e\right) = B_{k,t}, \quad \forall k = 1,\ldots,n, \tag{111}$$

$$\int_{(b_1,\ldots,b_k,\ldots,b_n,x,e)\in(\mathbb{R}^+)^{n+1}\times E} x_t\, d\Lambda_t\left(b_1,\ldots,b_k,\ldots,b_n,x,e\right) = V. \tag{112}$$

Finally, for a given stochastic process ε_t, an equilibrium of this economy is a set of shocks, labor market and public finance variables consistent with the processes (101)-(110), a set of policy rules $\left\{c_t^i,\, x_t^t,\, \left(b_{k,t}^i\right)_{k=1,\ldots,n},\, l_t^i\right\}_{t=0,\ldots,\infty}$ that solve the program of the agents ((95)-(100)), and prices $\{Q_t,\, (p_{t,k})_{k=1,\ldots,n}\}_{t=0,\ldots,\infty}$ that satisfy the market-clearing conditions (111)–(112).

2.1 The equilibrium

We now describe the equilibrium studied in Section **5** of the paper. We proceed by construction and follow a guess-and-verify strategy. We conjecture that agents face a binding borrowing constraint after two periods of unemployment (sufficient conditions for this to be the case are provided below). There are 5 different types of agents in this economy: *ee*, *ue*, *eu*, *euu* and *uuu*. First, agents *ee* and *ue* are currently employed and differ only with respect to their initial asset holdings and thus their current labor supply. Their consumption at date t is the same and denoted c_t^e. Both will participate in asset markets. Second, *eu* agents were employed at the previous period but are currently unemployed. They also participate to asset markets under the conjectured equilibrium (since it takes two periods of unemployment for the portfolio to be entirely liquidated). Third, *euu* and *uuu* are unemployed for at least two periods and will face binding borrowing constraints; as a consequence, they will not participate in asset markets.

From the general budget constraint of an agent, the consumption of e (i.e., both *ee* and *ue*), *eu*, *euu*, and *uuu* agents at date t can be expressed as follows:

$$c_t^e = u'^{-1}\left(1/z_t\right), \tag{113}$$

$$c_t^{eu} = \sum_{j=1}^{n} p_{t,j-1}\, b_{t-1,j}^e + \left(Q_t + y_t\right) x_{t-1}^e - \sum_{j=1}^{n} p_{t,j}\, b_{t,j}^{eu} - Q_t x_t^{eu} + \delta - \tau_t^u, \tag{114}$$

$$c_t^{euu} = \sum_{j=1}^{n} p_{t,j-1}\, b_{t,j}^{eu} + \left(Q_t + y_t\right) x_t^{eu} + \delta - \tau_t^u, \tag{115}$$

$$c_t^{uuu} = \delta. \tag{116}$$

Equation (113) follows from the first-order condition characterising the optimal labour supply of employed agents. The other consumption levels follow from agents' budget constraint and the fact that asset liquidation is partial in the first period of unemployment, and full in the second.

The bond Euler equations are given by, for $k = 1, \ldots, n$:[2]

$$\frac{p_{t,k}}{z_t} \geq \beta E_t \left[\alpha_{t+1} \frac{1}{z_{t+1}} + \beta E_t \left(1 - \alpha_{t+1}\right) u' \left(c_{t+1}^{eu}\right) \right] p_{t+1,k-1}, \tag{117}$$

$$p_{t,k} u' \left(c_t^{eu}\right) \geq \beta E_t \left[\left(1 - \rho_{t+1}\right) \frac{1}{z_{t+1}} + \rho_{t+1} u' \left(c_{t+1}^{euu}\right) \right] p_{t+1,k-1}, \tag{118}$$

where $E_t[.]$ is the expectation over aggregate risk. The Euler equations for the risky asset are given by:

$$\frac{Q_t}{z_t} \geq \beta E_t \left[\alpha_{t+1} \frac{1}{z_{t+1}} + \left(1 - \alpha_{t+1}\right) u' \left(c_{t+1}^{eu}\right) \right] \left(Q_{t+1} + y_{t+1}\right), \tag{119}$$

$$Q_t u' \left(c_t^{eu}\right) \geq \beta E_t \left[\left(1 - \rho_{t+1}\right) \frac{1}{z_{t+1}} + \rho_{t+1} u' \left(c_{t+1}^{euu}\right) \right] \left(Q_{t+1} + y_{t+1}\right). \tag{120}$$

Finally, the market-clearing conditions of this economy give:

$$\omega_t^e b_{t,k}^e + \omega_t^{eu} b_{t,k}^{eu} = B_{t,k}, \quad k = 1, \ldots, n, \tag{121}$$

$$\omega_t^e x_t^e + \omega_t^{eu} x_t^{eu} = V. \tag{122}$$

2.2 Conditions for the existence of an equilibrium with two-period liquidation

The conjectured equilibrium exists provided that *euu* agents face binding borrowing constraints on all assets. *uuu* agents (unemployed for three or more periods in a row) have zero beginning of period wealth and are thus poorer that *euu* agents. Hence, they will also face a binding borrowing constraint if *euu* do. A sufficient condition for *euu* agents to face binding borrowing constraints is:

$$p_{t,k} u' \left(c_t^{euu}\right) > \beta E_t \left[\left(1 - \rho_{t+1}\right) \frac{1}{z_{t+1}} + \rho_{t+1} u' \left(c_t^{uuu}\right) \right] p_{t+1,k-1}, \tag{123}$$

$$Q_t u' \left(c_t^{euu}\right) > \beta E_t \left[\left(1 - \rho_{t+1}\right) \frac{1}{z_{t+1}} + \rho_{t+1} u' \left(c_{t+1}^{uuu}\right) \right] \left(Q_{t+1} + y_{t+1}\right). \tag{124}$$

We infer that for a given stochastic process (ε_t), an equilibrium is the set of $5n + 20$ variables $\{\{p_{t,k}, Q_t, c_t^e, c_t^{eu}, c_t^{euu}, c_t^{uuu}, x_t^e, x_t^{eu}, b_{t,k}^e, b_{t,k}^{eu}, \omega_t^e, \omega_t^{ee}, \omega_t^{eu}, \omega_t^{uu}, \omega_t^{uuu}, \omega_t^{ue}, \tau_t^e, \tau_t^u, v_t, z_t, y_t, \alpha_t, \rho_t, A_{t,k}, B_{t,k}\}_{k=1,\ldots,n}^{t\in\mathbb{N}}$ for the $5n + 20$ equations (101)-(110) and (113)–(122), which satisfy the conditions (123)-(124).

[2] We use inequality conditions for Euler equations, as some agents may not hold all maturities. In this case, the Euler equation holds with equality only for the relevant maturities.

2.3 Summary of the baseline calibration

In the baseline calibration provided in the paper, agents eu hold only $1-$period bonds. That is equilibrium prices are such that the following relations holds:

$$x_t^{eu} = 0, \tag{125}$$

$$b_{t,k}^{eu} = 0, \quad k = 2, \dots, n, \tag{126}$$

$$p_{t,k} u'\left(c_t^{eu}\right) > \beta E_t \left[\left(1 - \rho_{t+1}\right) \frac{1}{z_{t+1}} + \rho_{t+1} u'\left(c_{t+1}^{euu}\right) \right] p_{t+1,k-1}, \quad k = 2, \dots, n, \tag{127}$$

$$Q_t u'\left(c_t^{eu}\right) > \beta E_t \left[\left(1 - \rho_{t+1}\right) \frac{1}{z_{t+1}} + \rho_{t+1} u'\left(c_{t+1}^{euu}\right) \right] \left(Q_{t+1} + y_{t+1}\right). \tag{128}$$

Furthermore, the Euler equations for employed agents (117)–(119) hold with equality for all $k = 1, \dots, n$, while only (118) with $k = 1$ holds with equality for eu agents.

We can summarize the equilibrium as follows:

$$
\begin{aligned}
c_t^e &= u'^{-1}\left(1/z_t\right), \\
c_t^{eu} &= \sum_{j=1}^{n} p_{t,j-1}\, b_{t-1,j}^e + \left(Q_t + y_t\right) x_{t-1}^e - p_{t,1}\, b_{t,1}^{eu} + \delta - \tau_t^u, \\
c_t^{euu} &= b_{t,1}^{eu} + \delta - \tau_t^u, \\
c_t^{uuu} &= \delta \\
\frac{p_{t,k}}{z_t} &= \beta E_t \left[\alpha_{t+1} \frac{1}{z_{t+1}} + \beta E_t \left(1 - \alpha_{t+1}\right) u'\left(c_{t+1}^{eu}\right)\right] p_{t+1,k-1}, k = 1, \ldots, n, \\
p_{t,1} u'\left(c_t^{eu}\right) &= \beta E_t \left[\left(1 - \rho_{t+1}\right) \frac{1}{z_{t+1}} + \rho_{t+1} u'\left(c_{t+1}^{euu}\right)\right], \\
\frac{Q_t}{z_t} &= \beta E_t \left[\alpha_{t+1} \frac{1}{z_{t+1}} + \left(1 - \alpha_{t+1}\right) u'\left(c_{t+1}^{eu}\right)\right] \left(Q_{t+1} + y_{t+1}\right), \\
b_{t,2}^{eu} &= \ldots = b_{t,n}^{eu} = x_t^{eu} = 0, \\
\omega_t^e b_{t,1}^e + \omega_t^{eu} b_{t,1}^{eu} &= \sum_{j=0}^{n-k} A_{t-j,1+j}, \\
\omega_t^e b_{t,k}^e &= \sum_{j=0}^{n-k} A_{t-j,k+j}, \quad k = 2, \ldots, n, \\
\omega_t^e x_t^e &= V, \\
\left(\omega_t^e + \omega_t^u \chi\right) \tau_t^e &= \sum_{j=1}^{n} A_{t-j,j} - \sum_{j=1}^{n} p_{t,j} A_{t,j}, \\
A_{t,k} &= A_k - \mu_k\, z_t, \quad k = 1, \ldots, n, \\
v\omega_t^e &= \omega_t^{ee} + \omega_t^{ue}, \\
\omega_t^u &= \omega_t^{eu} + \omega_t^{uu}, \\
\omega_t^{eu} &= \left(1 - \alpha_t\right)\left(\omega_{t-1}^{ee} + \omega_{t-1}^{ue}\right), \\
\omega_t^{ee} &= \alpha_t \left(\omega_{t-1}^{ee} + \omega_{t-1}^{ue}\right), \\
\omega_t^{uu} &= \rho_t \left(\omega_{t-1}^{eu} + \omega_{t-1}^{uu}\right), \\
\omega_t^{ue} &= \left(1 - \rho_t\right)\left(\omega_{t-1}^{eu} + \omega_{t-1}^{uu}\right),
\end{aligned}
$$

together with the shocks:

$$z_t = \exp(v_t) \quad \text{with} \quad v_t = \lambda_v v_{t-1} + \sigma_v \varepsilon_t,$$

$$\alpha_t = \bar{\alpha} + \sigma_\alpha \left(e^{v_t} - 1\right),$$

$$\rho_t = \bar{\rho} + \sigma_\rho \left(e^{v_t} - 1\right),$$

$$y_t = \sigma_y e^{v_t}.$$

The model is solved by a perturbation method. We first determine the steady state and check that it is well defined and that the required existence conditions hold. We then perform second-order expansions of all the models' equation around the steady state, and then run stochastic simulations on this approximate model.

3 Additional results

3.1 Robustness of our results to an alternative taxation scheme

Our analysis of the yield curve with positive net supply of bonds has been undertaken under the assumption that only employed agents pay taxes. This effectively insulates bondholders' pricing kernel from future taxes and their variability, and thereby allows us to isolate the liquidation risk premium on long bonds in a clear cut manner.[3] To see most clearly how the tax structure affects our results, assume instead that a lump-sum tax $\tilde{\tau}_t$ is levied on *all* agents symmetrically, including the unemployed. In this case the marginal utility term in (132) becomes

$$u'\left(\delta + \frac{1}{\omega^e}\left(\sum_{j=1}^{n} p_{t+1,j-1}\left(s^{t+1}\right) B_j\right) - \tilde{\tau}_{t+1}\left(s^{t+1}\right)\right), \tag{129}$$

where, by the government budget constraint, the lump sum tax charged to all agents is

$$\tilde{\tau}_t(s^t) = \sum_{k=1}^{n} \left(p_{t,k-1}(s^t) - p_{t,k}(s^t)\right) B_k, \tag{130}$$

which replaces (5) in the body of the paper.

Equation (130) shows that time-variations in bond prices induce variations in taxes, because they alter the borrowing rates faced by the government and hence the amount of taxes that is required to roll over a given maturity structure of the debt. When taxes enter the marginal utility associated with a bad idiosyncratic shock as in (129), their level and volatility affect the equilibrium pricing kernel and feed back to all bond prices and yields. In other words, taxes in the pricing kernel

[3]On the one hand, quasi-linear preferences imply that employed agents, who work as much as necessary to bring their marginal utility of wealth to $1/z$, fully neutralise the effect of taxes on their consumption. On the other hand, when the unemployed are not taxed, taxes do not affect the marginal utility of those agents who liquidate their bond portfolio. Taken together, these two features imply that taxes do not appear in bondholders' pricing kernel.

introduce a *refinancing risk* that adds up to the liquidation risk effects isolated above. Two features of this additional source of risk are worth mentioning. First, it is positively correlated with the liquidation risk, and hence magnifies the effects of the latter on the yield curve. Indeed, when the aggregate state is unfavourable –and consequently bond prices and the liquidation value of the portfolio are low–, then rolling over the debt is expensive and taxes are high, which further depresses the consumption of agents faced by a bad idiosyncratic shocks; conversely, high bond prices in the good aggregate state raise the liquidation value of the portfolio and, at the same time, lower taxes. It follows that the yield curve is more sensitive to changes in bond volumes when both employed and unemployed agents are taxed. Second, changes in the supply of one-period bonds do affect the refinancing risk –whereas they do not affect the liquidation risk. Indeed, while one-period bonds' payoff is noncontingent, the price at which they are issued is and thereby affects the amount of taxes required to maintain a given maturity structure of the debt.

None of the results in Section **4.1** (complete markets) and Section **4.2** (incomplete-markets/zero net bond supply) of the paper are modified when switching from the baseline taxation scheme to the uniform taxation scheme. Indeed, in the complete-market case, Ricardian Equivalence holds and hence the timing of taxes does not matter. In the incomplete-market/zero net supply case, taxes are zero at all times (by (130)), hence the way agents are taxed does not matter. Proposition 4 in the paper, which gathers our results in the incomplete-market/positive net supply case, is modified as follows:

Proposition 5 (Incomplete-market, positive volume yield curves) *When all agents are uniformly taxed, there exists a unique equilibrium such that:*

1. *All employed agents buy the same amount of bonds of each maturity, while all unemployed agents face a binding borrowing constraint (and consequently hold no bonds);*
2. *The date t price of a bond of maturity k is $\widetilde{p}_{t,k} = E_t\left[\widetilde{m}^{PV}_{t+1}\widetilde{p}_{t+1,k-1}\right] = E_t\prod_{j=1}^{k}\widetilde{m}^{PV}_{t+j}$, where*

$$\widetilde{m}^{PV}_{t+1} = m^{CM}_{t+1}\widetilde{I}^{PV}_{t+1} \quad \text{and} \tag{131}$$

$$\widetilde{I}^{PV}_{t+1} \equiv \frac{\alpha/z_{t+1} + (1-\alpha)\,u'\left(\delta + \frac{1-\omega^e}{\omega^e}\sum_{j=1}^{n}\widetilde{p}_{t+1,j-1}\,B_j + \sum_{j=1}^{n}\widetilde{p}_{t+1,j}\,B_j\right)}{1/z_{t+1}}; \tag{132}$$

3. *An increase in the supply of bonds of any maturity raises both the level and the slope of the yield curve;*

There are two main differences between proposition 4 in the paper and proposition 5 above:

1. The pricing kernel, and more specifically its idiosyncratic component $\widetilde{I}_{t+1}^{PV}$, is modified because taxes now affect the marginal utility of agents who fall into unemployment. Since the taxes required to roll over a given maturity of the debt depend on bond prices (see (130) again), this introduces an additional term with bond prices in the pricing kernel (relative to the case where only employed agents are taxed).

2. The last statement of Proposition 4 regarding the particular role of 1-period bond does not hold any more. This is because the quantity of one-period bonds does affect taxes (via the refinancing of this short debt, see (130)), and hence directly enters bondholders' pricing kernel.

Except for these two modifications, none of our baseline results are changed. In particular, the equilibrium can be shown to exist under small bond volumes and small aggregate shocks, an the impact of bond volumes on the shape of the yield curve are similar (except again for point 4 of Proposition 4 in the paper).

3.2 Impact of bond volumes on welfare

3.2.1 Baseline taxation scheme

We analyze the welfare impact on each agent of changes in bond volumes in our baseline theoretical model.[4] For simplicity, we carry out this analysis in an economy without aggregate risk (i.e., $z^l = z^h = 1$), and in which idiosyncratic uncertainty is not time-varying (i.e., $\alpha^h = \alpha^l$). We then have the following proposition:

Proposition 6 (Bond supplies and welfare) *A greater supply of bonds:*

(i) always increases the welfare of agents who stay employed or fall into unemployment, but increases the welfare of agents who leave unemployment or stay unemployed if and only if $\beta > [\alpha + (1-\alpha)u'(\delta)]^{-1}$*;*

(ii) always increases the ex ante welfare (at date 0 *and before agents know their type).*

Proof.

[4] Aiyagari and McGrattan (1998) and Floden (2001) have offered quantitative assessments of the aggregate welfare effect of changes in the stock of debt in economies with incomplete markets for idiosyncratic risks.

Instantaneous utility. We define U as the vector of instantaneous utilities: $U = \left[u(c^k) - l^k\right]_{k=ee,ue,eu,uu}$. There is no time index since there is no aggregate risk. The price of a $k-$period bond is denoted p_k, quantity of bonds of maturity k in any bondholder's portfolio is simply B_k/ω^e (by the market clearing condition and symmetry of portfolios). From agents' budget constraints and optimality conditions (See Section 1 above), the realised consumption levels by agent types are given by:

$$c^{eu} = \delta + \sum_{k=1}^{n} p_{k-1} \frac{Bk}{\omega^e}, \qquad c^{uu} = \delta,$$
$$c^{ee} = c^{ue} = u'^{-1}(1).$$

The taxes paid by the (employed) agents are constant and equal to $\tau = \sum_{k=1}^{n} (p_{k-1} - p_k) \frac{B_k}{\omega^e}$. From agents consumption choices and budget constraints, we infer the labour supply choices of employed agents (ee and ue) to be:

$$l^{ee} = c^{ee} + \tau + \sum_{k=1}^{n} p_k \frac{B_k}{\omega^e} - \sum_{k=1}^{n} p_{k-1} \frac{B_k}{\omega^e} = u'^{-1}(1),$$
$$l^{ue} = c^{ee} + \tau + \sum_{k=1}^{n} p_k \frac{B_k}{\omega^e} = u'^{-1}(1) + \sum_{k=1}^{n} p_{k-1} \frac{B_k}{\omega^e}.$$

This gives the following vector of instant utility by agent types:

$$U = \begin{bmatrix} u(u'^{-1}(1)) - u'^{-1}(1) \\ u(u'^{-1}(1)) - u'^{-1}(1) - \sum_{k=1}^{n} p_{k-1} \frac{B_k}{\omega^e} \\ u\left(\delta + \sum_{k=1}^{n} p_{k-1} \frac{B_k}{\omega^e}\right) \\ u(\delta) \end{bmatrix}.$$

Since $p_1 = \beta\,(\alpha + (1-\alpha)u'(\delta))$, we have (cf. (50)) $p_k = p_1 p_{k-1}$. The derivative of the instant utility vector U with respect to bond supplies in the vicinity of the no-trade equilibrium is given by:

$$\left.\frac{\partial U}{\partial B_k}\right|_{B_k=0} = \frac{p_{k-1}}{\omega^e} \begin{bmatrix} 0 & -1 & u'(\delta) & 0 \end{bmatrix}^\top.$$

Intertemporal utility. We define $\mathcal{U}$ as the vector of the four individual intertemporal utilities: $\mathcal{U} = \sum_{k=0}^{\infty} \beta^k \, \Omega^k \, U = \sum_{k=0}^{\infty} \beta^k \, Q \, D^k \, Q^{-1} \, U$. The transition matrix Ω across the four possible agent types $\{ee\ ue\ eu\ uu\}$ is given by:

$$\Omega = \begin{bmatrix} \alpha & 0 & 1-\alpha & 0 \\ \alpha & 0 & 1-\alpha & 0 \\ 0 & 1-\rho & 0 & \rho \\ 0 & 1-\rho & 0 & \rho \end{bmatrix} = Q.D.Q^{-1}, \text{ with } Q = \begin{bmatrix} 1 & 1-\alpha & 0 & 1-\alpha \\ 1 & 0 & \rho & 1-\alpha \\ 1 & -\alpha & 0 & -(1-\rho) \\ 1 & 0 & -(1-\rho) & -(1-\rho) \end{bmatrix}, \tag{133}$$

and $D = \text{Diag}(1\ 0\ 0\ \alpha + \rho - 1)$. (134)

Ex post utility After some matrix manipulation, the impact of bond volumes on ex post utilities can be written as:

$$\frac{\partial \mathcal{U}}{\partial B_k} = \frac{p_{k-1}}{\omega^e(1-\beta)(1-\beta(\alpha+\rho-1))} \begin{bmatrix} \beta(1-\alpha)\left((1-\beta\rho)u'(\delta)-\beta(1-\rho)\right) \\ (1-\beta\rho)\left(\beta(\alpha+(1-\alpha)u'(\delta))-1\right) \\ (1-\beta\alpha)\left((1-\beta\rho)u'(\delta)-\beta(1-\rho)\right) \\ \beta(1-\rho)\left(\beta(\alpha+(1-\alpha)u'(\delta))-1\right) \end{bmatrix}.$$

This implies that: $\frac{\partial \mathcal{U}^{ee}}{\partial B_k}, \frac{\partial \mathcal{U}^{eu}}{\partial B_k} > 0$, but $\frac{\partial \mathcal{U}^{ue}}{\partial B_k}, \frac{\partial \mathcal{U}^{uu}}{\partial B_k} < 0$ if and only if $\beta < \beta^{\text{ex post}} = (\alpha + (1-\alpha)u'(\delta))^{-1}$.

Ex ante utility To compute the impact of bond volumes on ex ante welfare, we multiply the ex post utility vector by the weight vector $W = \frac{1}{2-\alpha-\rho}[\alpha(1-\rho),\ (1-\alpha)(1-\rho),\ (1-\alpha)(1-\rho),\ \rho(1-\alpha)]$. This gives:

$$W\frac{\partial \mathcal{U}}{\partial B_k} = \frac{p_{k-1}}{\omega^e}\frac{(1-\alpha)(1-\rho)(u'(\delta)-1)}{(1-\beta)(2-\alpha-\rho)} > 0,$$

which is always positive.■

To summarise, Proposition 6 compares the intertemporal welfare of the four agent types in two economies that marginally differ in their supply of bonds. In short, and using the same notation as in Section **4** of the paper, when the supply of bonds increases, the intertemporal welfare of *ee* and *eu* agents always does but the intertemporal welfare of *ue* or *uu* agents increases only if agents' subjective discount factor is sufficiently high. This is due to the impact of bond supplies on agents' ability to self-insure against idiosyncratic shocks. For agents who hold bonds at the beginning of the period (namely, *ee* and *eu* agents), these better self-insurance possibility are always beneficial. However, agents holding no asset at the beginning of the period (i.e. *ue* and *uu*) benefit form an increase in bond volumes only if they are sufficiently patient enough to value the gains from more (self-)insurance in the future. Obviously, these welfare effects are not independent of the tax structure, as we discuss further below.

3.2.2 Impact of the taxation scheme

We have discussed in Section 3.1 the impact of an alternative taxation scheme on the way bond supplies affect bondholders' pricing kernel and, by way of consequence, the shape of the yield curve. We now discuss how this scheme alters the welfare impact of bond volumes, relative to the baseline taxation scheme. The central difference is that unemployed agents (who now pay the tax) may now see their ex ante welfare fall after an increase in bond supplies. More precisely, we have the following proposition.

Proposition 7 (Bond supplies and welfare when unemployed are taxed) *A greater supply of bonds increases ex ante welfare (at date 0 and before agents know their type) if and only if* $\beta > \frac{\alpha+\rho-1}{\alpha+(1-\alpha)u'(\delta)}$. *The impact on the ex-post welfare is the same as in the previous tax system (See Proposition 6).*

Proof. The proof is very similar to that of the previous proposition, so we just provide the main steps.

Instantaneous utility. The consumption levels of the four agent types are:

$$
\begin{aligned}
c^{eu} &= \delta + \sum_{k=1}^{n} \left(\frac{1-\omega^e}{\omega^e} p_{k-1} + p_k \right) B_k, \qquad c^{uu} = \delta - \sum_{k=1}^{n} (p_{k-1} - p_k) B_k, \\
c^{ee} &= c^{ue} = u'^{-1}(1),
\end{aligned}
$$

while the labour supply levels of employed agents are given by:

$$
\begin{aligned}
l^{ee} &= u'^{-1}(1) + \frac{1-\omega^e}{\omega^e} \sum_{k=1}^{n} (p_k - p_{k-1}) B_k, \\
l^{ue} &= u'^{-1}(1) + \sum_{k=1}^{n} \left(p_{k-1} + \frac{1-\omega^e}{\omega^e} p_k \right) B_k.
\end{aligned}
$$

From these consumption and labour supply choices, we infer the vector of instant utility by agent types to be:

$$
U = \begin{bmatrix}
u(u'^{-1}(1)) - u'^{-1}(1) - \frac{1-\omega^e}{\omega^e} \sum_{k=1}^{n} (p_k - p_{k-1}) B_k \\
u(u'^{-1}(1)) - u'^{-1}(1) - \sum_{k=1}^{n} (p_{k-1} + p_k \frac{1-\omega^e}{\omega^e}) B_k \\
u(\delta + \sum_{k=1}^{n} (p_{k-1} \frac{1-\omega^e}{\omega^e} + p_k) B_k) \\
u(\delta - \sum_{k=1}^{n} (p_{k-1} - p_k) B_k)
\end{bmatrix},
$$

while its derivative w.r.t. bond volumes is

$$
\frac{\partial U}{\partial B_k} = p_{k-1} \begin{bmatrix}
-\frac{1-\omega^e}{\omega^e}(p_1 - 1) \\
-(1 + \frac{1-\omega^e}{\omega^e} p_1) \\
(\frac{1-\omega^e}{\omega^e} + p_1) u'(\delta) \\
-(1 - p_1) u'(\delta)
\end{bmatrix}.
$$

Ex post utility The impact of bond volumes on ex post utilities is now given by

$$
\frac{\partial \mathcal{U}}{\partial B_k} = p_{k-1} \begin{bmatrix}
\frac{1-\omega^e}{\omega^e} + \frac{\beta^2(1-\alpha)(u'(\delta)-1)(1-\rho+(1-\alpha)u'(\delta))}{(1-\beta)(1-\beta(\alpha+\rho-1))} \\
- \frac{(1-\beta\,(\alpha+u'(\delta)\,(1-\alpha)))(1-\beta\rho+\beta(1-\alpha)u'(\delta))}{(1-\beta)(1-\beta(\alpha+\rho-1))} \\
\frac{1-\omega^e}{\omega^e} u'(\delta) + \frac{\beta(1-\alpha\beta)(u'(\delta)-1)(1-\rho+(1-\alpha)u'(\delta))}{(1-\beta)(1-\beta(\alpha+\rho-1))} \\
- \frac{(1-\beta\,(\alpha+(1-\alpha)u'(\delta)))(\beta(1-\rho)+(1-\beta\alpha)u'(\delta))}{(1-\beta)(1-\beta(\alpha+\rho-1))}
\end{bmatrix}.
$$

We find that $\frac{\partial \mathcal{U}^{ee}}{\partial B_k}, \frac{\partial \mathcal{U}^{eu}}{\partial B_k} > 0$, but $\frac{\partial \mathcal{U}^{ue}}{\partial B_k}, \frac{\partial \mathcal{U}^{uu}}{\partial B_k} < 0$ if and only if $\beta < \beta^{\text{ex post}} = (\alpha + u'(\delta)\ (1-\alpha))^{-1}$.

Ex ante utility Again, premultiplying $\frac{\partial \mathcal{U}}{\partial B_k}$ by the appropriate weight vector, we get:

$$W\frac{\partial \mathcal{U}}{\partial B_k} = p_{k-1}\,(1-\alpha)\frac{(1-\beta+\beta u'(\delta))(\alpha+(1-\alpha)u'(\delta)) - (1-\rho+\rho\, u'(\delta))}{(1-\beta)(2-\alpha-\rho)},$$

which is negative whenever $\beta < \beta^{\text{ex ante}} = \frac{\alpha+\rho-1}{\alpha+(1-\alpha)u'(\delta)} (< \beta^{\text{ex post}})$.

3.3 Rejection of the Expectation Hypothesis

The Expectation Hypothesis states that the excess return on long term bonds over short term ones cannot be predicted. Up to a constant risk premium which may depend on the maturity, the excess return is zero on average. However, many empirical studies (Campbell and Shiller (1991) for nominal bonds and Pflueger and Viceira (2011) for real bonds among many others) have shown that the yield spread between long and short bonds was a robust predictor of the excess return on long bonds. In our economy, the spread helps forecast excess returns as long as the aggregate state is persistent ($\pi^h + \pi^l > 1$) and the transition rates across aggregate states are not identical ($\pi^h \neq \pi^l$).

For simplicity, we focus on the predictability of excess returns on two period bonds, for which we have the following Proposition:

Proposition 8 (Rejection of the Expectation Hypothesis) *We define the expected excess return y_2^s in state $s = h, l$ for a 2 period bond and the corresponding spread δr_2^s as follows:*

$$y_2^s = \ln\left(\frac{E_s\left[p_{1,t+1}\right]}{p_2^s}\right)$$
$$\delta r_2^s = r_2^s - r_1^s$$

The Expectation Hypothesis is rejected – and more precisely the regression of y_2^s over δr_2^s generates a coefficient $\widehat{\beta}$ different from zero – as long as aggregate states are not iid $\pi^h + \pi^l \neq 1$.

Proof of Proposition 8

As in Proposition 3 for example, we assume a mean preserving spread on the aggregate risk $z^h = z(1 + 2(1-q)\varepsilon)$ and $z^l = z(1 - 2q\varepsilon)$ where z is the unconditional average, ε the shock.

We need to express the excess return $y_2^s - r_1^s$ and the spread $r_2^s - r_1^s$ in both states $s = h, l$. We compute respectively second and first order Taylor developments and, we obtain:

$$y_2^s - r_1^s = \ln\left(\frac{p_1^s E_s\left[p_{1,t+1}\right]}{p_2^s}\right) = \ln\left(\frac{C_1^s E_s\left[p_{1,t+1}\right]}{C_2^s}\right) = 4\pi^s(1-\pi^s)\frac{\alpha\left(\alpha(2-\pi^h-\pi^l) + (1-\alpha)zu'(\delta)\right)}{(\alpha+(1-\alpha)zu'(\delta))^2}\varepsilon^2, \quad (135)$$

$$\frac{r_2^h - r_1^h}{1-\pi^h} = -\frac{r_2^l - r_1^l}{1-\pi^l} = \frac{\alpha(2-\pi^h-\pi^l) + (1-\alpha)zu'(\delta)}{\alpha+(1-\alpha)zu'(\delta)}\varepsilon. \quad (136)$$

We begin with proving (135), which is equivalent to:

$$y_2^s - r_1^s = \ln\left(\frac{p_1^s E_s\left[p_{1,t+1}\right]}{p_2^s}\right) = \ln\left(\frac{C_1^s E_s\left[p_{1,t+1}\right]}{C_2^s}\right). \tag{137}$$

Expression of the numerator in (137). We already know from (21) and (22) that we have:

$$\begin{aligned}
\frac{zC_1^{ZV,h}}{\beta} &= \alpha + (1-\alpha)u'(\delta) - 2\alpha\frac{1-\pi^h}{2-\pi^h-\pi^l}\left(\pi^h+\pi^l-1-2\frac{(1-\pi^h)(\pi^h+\pi^l-1)+(1-\pi^l)(2-\pi^h-\pi^l)}{2-\pi^h-\pi^l}\varepsilon\right)\varepsilon,\\
\frac{E_h p_1^{ZV}}{\beta} &= \alpha + (1-\alpha)zu'(\delta)\\
&\quad + 2\frac{(1-\pi^h)(\pi^h+\pi^l-1)}{2-\pi^h-\pi^l}\Big((2-\pi^h-\pi^l)\alpha + (1-\alpha)zu'(\delta)\Big)\varepsilon + 4\alpha\frac{(1-\pi^h)(1-\pi^l)}{2-\pi^h-\pi^l}\varepsilon^2.
\end{aligned}$$

So, we obtain:

$$\begin{aligned}
\frac{zC_1^{ZV,h}}{\kappa\beta}\frac{E_h p_1^{ZV}}{\kappa\beta} &= 1 + 2(1-\pi^h)(\pi^h+\pi^l-1)\Omega\Big(\kappa - (\pi^h+\pi^l)\alpha\Big)\varepsilon\\
&\quad + 4\alpha\frac{(1-\pi^h)}{2-\pi^h-\pi^l}\Big((1-\pi^h)(\pi^h+\pi^l-1) + 2(1-\pi^l)(2-\pi^h-\pi^l)\Big)\Omega\varepsilon^2\\
&\quad - 4\alpha(1-\pi^h)^2(\pi^h+\pi^l-1)^2\Omega^2\Big((2-\pi^h-\pi^l)\alpha + (1-\alpha)zu'(\delta)\Big)\varepsilon^2.
\end{aligned} \tag{138}$$

Expression of the denominator in (137). From (51), we have for $C_2^{ZV,h}$:

$$\begin{aligned}
\frac{C_2^{ZV,h}}{\beta} &= \pi^h\kappa^h C_1^{ZV,h} + (1-\pi^h)\kappa^l C_1^{ZV,l}\\
&= \kappa\Big(\pi^h C_1^{ZV,h} + (1-\pi^h)C_1^{ZV,l}\Big) + \frac{2(1-\alpha)(1-\pi^h)u'(\delta)}{2-\pi^h-\pi^l}\Big(\pi^h C_1^{ZV,h} - (1-\pi^l)C_1^{ZV,l}\Big)\varepsilon.
\end{aligned}$$

Therefore:

$$\begin{aligned}
\frac{zC_2^{ZV,h}}{\beta^2} &= \kappa^2 - 2\kappa\alpha\frac{(1-\pi^h)}{2-\pi^h-\pi^l}\left(\Big(\pi^h+\pi^l-1\Big)^2 - 2\frac{\big(\pi^h(1-\pi^h)+(1-\pi^l)^2\big)(\pi^h+\pi^l-1) + \big(1-\pi^l\big)(2-\pi^h-\pi^l)}{2-\pi^h-\pi^l}\varepsilon\right)\varepsilon\\
&\quad + \frac{2(1-\pi^h)(\pi^h+\pi^l-1)(1-\alpha)u'(\delta)}{2-\pi^h-\pi^l}\kappa\varepsilon - \frac{4\alpha(1-\alpha)(1-\pi^h)(\pi^h+\pi^l-1)u'(\delta)}{(2-\pi^h-\pi^l)^2}\Big(\pi^h(1-\pi^h)+(1-\pi^l)^2\Big)\varepsilon^2\\
&= \kappa^2 - 2\kappa\frac{(1-\pi^h)(\pi^h+\pi^l-1)}{2-\pi^h-\pi^l}\Big(\alpha\Big(\pi^h+\pi^l\Big) - \kappa\Big)\varepsilon\\
&\quad + 4\alpha\frac{(1-\pi^h)}{(2-\pi^h-\pi^l)^2}\Big(\alpha\Big(\pi^h(1-\pi^h)+(1-\pi^l)^2\Big)(\pi^h+\pi^l-1) + \kappa(1-\pi^l)(2-\pi^h-\pi^l)\Big)\varepsilon^2,
\end{aligned}$$

and:

$$\begin{aligned}
\left(\frac{zC_2^{ZV,h}}{\kappa^2\beta^2}\right)^{-1} &= 1 + 2(1-\pi^h)(\pi^h+\pi^l-1)\Omega\Big(\alpha\Big(\pi^h+\pi^l\Big) - \kappa\Big)\varepsilon\\
&\quad - 4\alpha(1-\pi^h)\Omega^2\Big(\alpha\Big(\pi^h(1-\pi^h)+(1-\pi^l)^2\Big)(\pi^h+\pi^l-1) + \kappa(1-\pi^l)(2-\pi^h-\pi^l)\Big)\varepsilon^2\\
&\quad + 4(1-\pi^h)^2(\pi^h+\pi^l-1)^2\Omega^2\Big(\alpha\Big(\pi^h+\pi^l\Big) - \kappa\Big)^2\varepsilon^2.
\end{aligned} \tag{139}$$

Expression of (137). Putting together (138) and (139), we obtain:

$$\frac{zC_1^{ZV,h}}{\kappa\beta}\frac{E_h p_1^{ZV}}{\kappa\beta}\left(\frac{zC_2^{ZV,h}}{\kappa^2\beta^2}\right)^{-1} = 1 + 4\alpha\pi^h(1-\pi^h)(2-\pi^h-\pi^l)^2\Omega^2\Big((2-\pi^h-\pi^l)\alpha + (1-\alpha)zu'(\delta)\Big).$$

Taking the logarithm, we obtain the expression 135 of the spread $y_2^h - r_1^h$.

Expression of (136). From (23), we deduce the expression of $r_1^{ZV,h}$ at the first-order:

$$r_1^{ZV,h} = -\ln\left(\beta\alpha + \beta(1-\alpha)zu'(\delta)\right) - 2(1-\pi^h)\Omega\left((2-\pi^h-\pi^l)\alpha + (1-\alpha)zu'(\delta)\right)\varepsilon.$$

We deduce the expression of $r_2^{ZV,h} = -\frac{1}{2}\ln\left(C_2^h z^h\right)$ at the first-order from (139):

$$r_2^{ZV,h} = -\ln\left(\beta\alpha + \beta(1-\alpha)zu'(\delta)\right) - (1-\pi^h)\Omega(\pi^h+\pi^l)\left((2-\pi^h-\pi^l)\alpha + (1-\alpha)zu'(\delta)\right)\varepsilon$$

From both above equalities, we obtain:

$$r_2^{ZV,h} - r_1^{ZV,h} = (1-\pi^h)\Omega(2-\pi^h-\pi^l)\left((2-\pi^h-\pi^l)\alpha + (1-\alpha)zu'(\delta)\right)\varepsilon,$$

which is expression (136).

Expression of the OLS estimator $\widehat{\beta}$. The OLS estimator $\widehat{\beta}$, which is equal to the unconditional covariance of excess return and spread divided by the unconditional variance of the spread, expresses as:

$$\widehat{\beta} = \frac{Cov(y_2 - r_1, r_2 - r_1)}{V[r_2 - r_1]} = \frac{E[(y_2 - r_1)(r_2 - r_1)] - E[y_2 - r_2]E[r_2 - r_1]}{E[(r_2 - r_1)^2] - E[r_2 - r_1]^2}.$$

After substitution of (135) and (136), we obtain:

$$\widehat{\beta} = \frac{4\alpha(\pi^l - \pi^h)(\pi^h + \pi^l - 1)}{(2-\pi^h-\pi^l)\left(\alpha + (1-\alpha)zu'(\delta)\right)}\varepsilon.$$

This coefficient is different from zero as soon as aggregate states are persistent and $\pi^h \neq \pi^l$. However, in that case, it is possible to go one additional order in the Taylor development and obtain ($\pi = \pi^h = \pi^l$):

$$\widehat{\beta} = \frac{4\alpha\pi(1-2\pi)\left(\alpha(1-2\pi) + (1-\alpha)zu'(\delta)\right)}{\left(\alpha + (1-\alpha)zu'(\delta)\right)^2}\varepsilon^2,$$

which is different from zero as soon as aggregate states are not persistent. □

3.4 Economy stationary in growth rates

In the particular case where $u(c) = \ln c$, our production economy is stationary in growth rates if we make the following additional two assumptions:

1. bond volumes grow at the same rate as productivity z_t. More specifically, at any date t, the supply of bond with maturity j is $B_j z_t$;
2. home production at date $t+1$ is equal to δz_t.[5]

[5] This particular timing assumption regarding the stochastic trend in home production allows us to obtain bond

Define $g_{t+1} = z_{t+1}/z_t$. Using these assumptions, it is straightforward to derive the following Euler equation for a k-period bond:

$$p_{t,k} = \beta E_t \left[\left(\alpha + (1-\alpha) \frac{1}{\frac{\delta}{g_{t+1}} + \sum_{j=1}^{n} \frac{p_{t+1,j-1}}{g_{t+1}} B_j/\omega^e} \right) \frac{p_{t+1,k-1}}{g_{t+1}} \right],$$

while the corresponding Euler equation in the baseline level-stationary model was:

$$\frac{p_{t,k}}{z_t} = \beta E_t \left[\left(\alpha + (1-\alpha) \frac{1}{\frac{\delta}{z_{t+1}} + \sum_{j=1}^{n} \frac{p_{t+1,j-1}}{z_{t+1}} B_j/\omega^e} \right) \frac{p_{t+1,k-1}}{z_{t+1}} \right].$$

Both expressions are very similar. Similarly, the conditions for all unemployed agents to be borrowing-constrained becomes, for $k = 1, \ldots, n$:

$$\frac{g_t p_{t,k}}{\delta + \sum_{j=1}^{n} p_{t,j-1} \frac{B_j}{\omega^e}} > \beta(1-\rho) E_t \left[\frac{p_{t+1,k-1}}{g_{t+1}} \right] + \beta \rho E_t \left[\frac{p_{t+1,k-1}}{\delta} \right],$$

while the corresponding conditions in the body of the paper were:

$$\frac{p_{t,k}}{\delta + \sum_{j=1}^{n} p_{t,j-1} \frac{B_j}{\omega^e}} > \beta(1-\rho) E_t \left[\frac{p_{t+1,k-1}}{z_{t+1}} \right] + \beta \rho E_t \left[\frac{p_{t+1,k-1}}{\delta} \right].$$

Again, both expressions are very similar. The properties of equilibria in the two economies are very similar. More precisely, although the shapes of the yield curves differ in general, the qualitative effects of an increase in the supply of bonds are the same in the two economies.

3.5 Alternative specification for the borrowing constraint

In this section, we construct a finite state space equilibrium with an alternative specification for the borrowing constraints. Namely, we relax the debt limit on each maturity, but impose that *total* wealth to be non-negative. This alternative form of the borrowing constraints allows agents to be leveraged –i.e., to issue bonds at some maturity in order to buy bonds at other maturities– in order to improve consumption smoothing. For the sake of simplicity, we focus on the case with two maturities. Hence, every agent faces the nonnegative wealth constraint $\sum_{k=1}^{2} p_{t,k}\, b_{t,k}^i \geq 0$, and in addition $b_{t,k}^i \geq -\underline{b}$, with $\underline{b} \geq 0$, for $k = 1, 2$ (note that this set of constraints nests our baseline case where $b_{t,1}^i, b_{t,2}^i \geq 0$). A debt limit on individual maturities (in additional to the wealth constraint) will allow us to maintain the property that a finite state space can characterize the equilibrium (just as it did in the baseline theoretical model), because it will ensure some homogeneity across equilibrium portfolios. To see this, imagine that only the nonnegative wealth constraint applied, while every agent for whom it would bind would be free to choose an interior portfolio composition

Euler equations that look very similar to those in the baseline level-stationary model, but is by no means essential for the results.

–i.e., any $(b_{t,1}, b_{t,2})$ consistent with $\sum_{k=1}^{2} p_{t,k}\, b_{t,k}^i \geq 0$. It follows that an agent facing a binding constraint on total wealth at date t would choose an interior (i.e., unconstrained) end-of-period portfolio (which would in general differ from $(b_{t,1}, b_{t,2}) = (0, 0)$). But then, the realisation of the date $t+1$ aggregate state would impact total asset income at the beginning of date $t+1$, and thereby the optimal end-of-period $t+1$ portfolio along an interior solution, and so on. It follows that even for an agent facing a binding nonnegative wealth constraint all along, the entire history of *aggregate states* in general matters for the determination of the optimal portfolio; and since the overall constraint may be binding for an arbitrary number of periods, there is in general an infinite number of agents to follow in order to characterise the equilibrium. Here, a constraint on individual maturities serves the purpose of limiting the dependence of the portfolio composition on the aggregate state for those agents for whom the wealth constraint binds. Indeed, provided that the same maturity-specific constraint binds for all these agents (which will indeed be the case in the equilibrium under consideration), then equilibrium portfolios are corner and thus identical across such agents, regardless of the history of aggregate states. To summarise, the combination of the two types of constraints allows us to study leverage while maintaining tractability –via the existence of an equilibrium with finite state space.

Agents now face the following relaxed problem:

$$\max_{\{c^i, l^i, b^i\}} E_0^i \sum_{t=0}^{\infty} \beta^t \left(u\left(c_t^i\right) - l_t^i\right) \tag{140}$$

$$\text{s.t.} \quad c_t^i + \tau_t e_t^i + \sum_{k=1}^{2} p_{t,k}\, b_{t,k}^i = \sum_{k=1}^{2} p_{t,k-1}\, b_{t-1,k}^i + e_t^i z_t l_t^i + \left(1 - e_t^i\right)\delta,$$

$$\sum_{k=1}^{2} p_{t,k}\, b_{t,k}^i \geq 0, \tag{141}$$

$$b_{t,k}^i \geq -\underline{b}, \;\; k = 1, 2 \tag{142}$$

$$c_t^i, \;\; l_t^i \geq 0, \tag{143}$$

$$\lim_{t \to \infty} \beta^t u'\left(c_t^i\right) b_{t,k}^i = 0, \quad \text{for } k = 1, 2 \tag{144}$$

All the other equations defining the equilibrium are the same as in baseline theoretical model. In particular, the market-clearing conditions are

$$\int_{\left(b_{t,1}^i, b_{t,2}^i, e\right) \in (\mathbb{R}^+)^2 \times E} b_{t,k}^i \, d\Lambda_t\left(b_{t,1}^i, b_{t,2}^i, e\right) = B_k, \quad k = 1, 2. \tag{145}$$

For the sake of clarity, we call (140)-(144) the *relaxed* problem, and the *restricted* problem a similar problem where the borrowing constraints are as in the paper:

$$b^i_{t,k} \geq 0,\ \ k = 1, 2 \tag{146}$$

instead of the constraints (141)-(142).

Denote μ^i_t the Lagrange multiplier associated with (141) and ν^i_{kt} the Lagrange multiplier for the condition $b^i_{t,k} \geq -\underline{b}$, $k = 1, 2$. The optimality conditions are:

$$\begin{cases} u'\left(c^i_t\right) = 1/z_t & \text{if} \quad e^i_t = 1, \\ l^i_t = 0 & \text{if} \quad e^i_t = 0, \end{cases} \tag{147}$$

$$\left(u'\left(c^i_t\right) - \mu^i_t\right) p_{t,1} = \beta E_t u'\left(c^i_{t+1}\right) + \nu^i_{1t}. \tag{148}$$

$$\left(u'\left(c^i_t\right) - \mu^i_t\right) p_{t,2} = \beta E_t \left[u'\left(c^i_{t+1}\right) p_{t+1,1}\right] + \nu^i_{2t} \tag{149}$$

We conjecture (and then verify) the existence of an equilibrium with limited cross-sectional heterogeneity wherein

$$e^i_t = 1 \Rightarrow \mu^i_t = \nu^i_{1,t} = \nu^i_{2,t} = 0 \qquad \text{and} \qquad e^i_t = 0 \Rightarrow \mu^i_t > 0, \tag{150}$$

and

$$e^i_t = 0 \Rightarrow \nu^i_{1,t} > 0,\ \text{and}\ \nu^i_{2,t} = 0. \tag{151}$$

To summarise, we conjecture and verify the existence of an equilibrium in which i) the overall nonnegative wealth constraint binds for the unemployed but not for the employed, ii) none of the maturity-specific constraints binds for the employed, and iii) the constraint on one-period bonds binds for the unemployed, but that on two-period bonds does not (i.e., the unemployed have a short position in one-period bonds and a long position on two-period bonds). The reason for the latter property is that long bonds pay a higher yield than short bonds, and this spread is not arbitraged away by unemployed agents due to the maturity-specific constraint. The optimality conditions (150) and (151) together with the market-clearing conditions give the following portfolio compositions:

$$b^u_{t,1} = -\underline{b},\ b^u_{t,2} = \frac{p_{t,1}}{p_{t,2}}\underline{b}$$

$$b^e_{t,1} = \frac{B_1 + \omega^u \underline{b}}{\omega^e},\ b^e_{t,2} = \frac{B_2 - \omega^u \frac{p_{1t}}{p_{2t}}\underline{b}}{\omega^e}$$

A key difference with the restricted model is that the portfolios of all agents now depend on equilibrium bond prices, because the latter affect the ability of the unemployed to issue one-period bonds. Prices themselves depend on relative bond demands and hence on portfolio choices. The solution to the model is the solution this fixed-point problem. Substituting the consumption levels of eu agents into the bond Euler equations for employed agents, one obtains:

$$\frac{p_{1t}}{z_t} = \alpha\beta E_t \frac{1}{z_{t+1}} + (1-\alpha)\beta E_t \left[u'(\delta + \frac{B_1 + \eta^u \underline{b}_1}{\omega^e} + p_{1t+1} \frac{B_2 - \omega^u \frac{p_{1t}}{p_{2t}} \underline{b}_1}{\omega^e}) \right]$$

$$\frac{p_{2t}}{z_t} = \alpha\beta E_t \frac{p_{1t+1}}{z_{t+1}} + (1-\alpha)\beta E_t \left[p_{1t+1} u'(\delta + \frac{B_1 + \eta^u \underline{b}_1}{\omega^e} + p_{1t+1} \frac{B_2 - \eta^u \frac{p_{1t}}{p_{2t}} \underline{b}_1}{\omega^e}) \right]$$

As before we assume that $z_t \in \{z^l, z^h\}$, with the same transition matrix as in the restricted model. If the equilibrium exists, it has a finite state space summarised by the amounts of debt and asset holdings computed above. Once equilibrium prices are computed, it is easy to recover the implied individual consumptions levels and to verify that conditions (150)-(151) hold, as initially conjectured.

We solve the fixed point problem associated with the relaxed model numerically, using the same parameter values as those used in the numerical application of the restricted model (see Section 4.3 of the main paper). In addition, we here set the maturity-specific constraint to $\underline{b} = 0.002$. We again start with the zero net supply benchmark ($B_1 = B_2 = 0$) and then increase bond supplies to $B_1 = B_2 = 0.00273$. This change in the supply of government bonds has been re-calibrated so as to generate exactly the same change in the level of the yield curve as that in the restricted model. The table below reports the change in the level and the slope of the yield curve (measured by $r_2 - r_1$). An increase in bond volumes raises both the level and the slope, as found in the main paper.

Interest rates	r_1	r_2	$r_2 - r_1$
Benchmark economy (%)	1.800	2.270	0.469
Economy with higher debt (%)	1.836	2.306	0.470
Change in interest rates (bp)	3.61	3.64	0.03

Effect of a debt increase on the yield curve

The analysis above shows (i) that the equilibrium with limited cross-sectional heterogeneity that we propose in the main paper can be generalised to other environments –including environments in which "inside" and "outside" liquidity instruments co-exist at various maturities; and (ii) that the effects of bond supplies on the shape of the yield curve uncovered in the main paper hold under more relaxed forms of the borrowing constraint. We leave the full investigation of the relaxed model with endogenous leverage for future work.

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