Incomplete Markets and Derivative Assets

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Abstract

We analyze derivative asset trading in an economy in which agents face both aggregate and uninsurable idiosyncratic risks. Insurance markets are incomplete for idiosyncratic risk and, possibly, for aggregate risk as well. However, agents can exchange insurance against aggregate risk through derivative assets such as options. We present a tractable framework, which allows us to characterize the extent of risk-sharing in this environment. We show that incomplete insurance markets can explain some properties of the volume of traded derivative assets, which are difficult to explain in complete market economies.

Keywords: Incomplete markets, heterogeneous-agent models, imperfect risk sharing, derivative assets.

JEL codes: G1, G12, E44.

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1 Introduction

The pricing and allocation of derivative assets (such as options) have not been analyzed in infinite-horizon incomplete market economies, although these environments could help to explain some properties of derivative asset prices and traded volumes, which are difficult to rationalize in complete market environments. For instance, the traded volumes of derivative assets exhibit some correlation with aggregate risk: the greater the aggregate uncertainty, the higher the volatility of asset prices and the larger the traded volumes of options. This stylized fact is reported for instance in **?** and in **?**. This pattern for the volume of derivative assets cannot be justified in models where markets are complete before the introduction of derivative assets. [1] In those setups with redundant derivative securities, the volume of traded assets does not play any role. Models with incomplete insurance markets for idiosyncratic risks are obvious candidates to explain these properties. Indeed, agents facing different exposures to uninsurable risk may value aggregate risk differently and thus may be willing to exchange aggregate risk, as shown by **?** in a two-period economy

The goal of this paper is to analyze theoretically the prices and allocations of derivative assets for the aggregate risk, in an infinite-horizon incomplete market economy. Our analysis is based on a methodological contribution that enables us to prove the existence of the equilibrium and to characterize it in an environment featuring aggregate and idiosyncratic risks simultaneously. Our equilibrium relies on the assumption that only a small volume of assets is available for agents to self-insure themselves. This *small-trade* equilibrium allows for theoretical investigation with both positive trades and aggregate shocks. It has been used to study the yield curve in Challe, Le Grand and Ragot (**?**). Here, we extend the analysis to derivative assets by enabling agents to face different exposures to individual risks. No-trade equilibrium models, as in Constantinides and Duffie (**?**) or in **?**, belong to another class of incomplete market models allowing for theoretical investigation. However, these no-trade equilibrium models are (by construction) not well-suited for analysing the volume of traded options.

The type of derivative assets we consider are option contracts contingent on aggregate risk. Two reasons motivate our choice. First, Ross (**?**) showed that options contribute to complete markets for the exchange of aggregate risk and are thus efficient risk-sharing instruments. Second, options are one of the most common insurance contracts against aggregate risk in actual financial markets. Thus, the empirical properties of these securities are well-established. The analysis could easily be extended to other derivative assets.

We characterize the price and allocation of options in this economy. We show that derivative assets are traded because the valuation of aggregate risk differs across agents,

[1] In this framework, derivative prices can be determined using portfolio replication, as pionnered in the seminal contributions of Black and Scholes (**?**) and Merton (**?**).

generating variations in the volume of traded derivative assets which are consistent with the data. Our simple model allows us to identify various mechanisms that determine risk-sharing for the aggregate risk. We consider our paper as a complement to quantitative works in more general environments. As these environments are not tractable (since they generate a large amount of heterogeneity across agents), they have to be simulated using computational methods (see Krusell and Smith, **?** among many others). They are thus less transparent in terms of identification of the underlying mechanisms. Our paper also contributes to the equilibrium option pricing literature. Besides market incompleteness already mentioned above, two other reasons, which are heterogeneous preferences or heterogeneous beliefs, can be found in the literature to justify option trading.[2]

Finally, it is noteworthy that the irrelance result of Krueger and Lustig (**?**) does not apply in our setup. Krueger and Lustig have characterized conditions for which incomplete insurance markets do not matter for asset pricing. This irrelevance result relies on the assumption that the marginal utility of all agents is homothetic with the same degree, which does not hold in our setup. Indeed, agents are endowed with different marginal utilities that are not necessary homogeneous, because agents have access to production technologies depending on their (uninsurable) idiosyncratic status.

The remainder of the paper is organized as follows. Section 2 presents the environment and Section 3 describes our equilibrium. Section 4 gathers our results on prices and volumes of derivative assets. Section 5 concludes.

2 The environment

2.1 Risks and securities

We consider a discrete-time economy populated by two types of ex-ante infinitely-lived agents (the heterogeneity will be made clearer later). Each population $i = 1, 2$ is distributed on a segment J_i of size 1 according to a non-atomic measure.[3]

2.1.1 Aggregate risk and asset structure

Agents can trade shares of a Lucas tree, whose mass is V. At any date t, the price of one unit of the tree is P_t. The tree dividend payoff $(y_t)_{t\geq 0}$ is stochastic and constitutes the single aggregate shock of the economy. This aggregate shock, which can take n different

[2]See for heterogeneous preferences: Bates (**?**); Bhamra and Uppal (**?**); Grossman and Zhou (**?**); Weinbaum (**?**); Bongaerts, De Jong and Driessen (**?**). See for heterogeneous beliefs: Biais and Hillion (**?**); Buraschi and Jiltsov (**?**).

[3]This formulation is introduced to solve issues when applying the law of large numbers to a continuum of random variables (e.g., Feldman and Gilles (**?**) and Green (**?**)). From now on, we assume that the law of large numbers applies.

values $\{y_1, \dots, y_n\}$, evolves as a time-homogeneous Markov chain. Transition probabilities of moving from state k to state l $(k, l = 1, \dots, n)$ are denoted by π_{kl}.

In addition to shares of the Lucas tree, agents can write contracts conditional on the realization of the aggregate shock. More specifically, agents can trade H call options, whose payoff depends on the (endogenous) price P_t of the risky asset.[4] The option $h = 1, \dots, H$ is characterized by a strike price or exercise price (in short, a strike), which depends on the state of the economy and is denoted by $K^h(y_t)$. The option h purchased in period t with the price Q_t^h, pays off in period $t+1$ the maximum of zero and the difference between the asset price at $t+1$, P_{t+1}, and the strike $K^h(y_t)$, i.e., $\left(P_{t+1} - K^h(y_t)\right)^+$ (where $t \in \mathbb{R} \mapsto t^+ = \max(t, 0)$).

Since the strike can depend on the aggregate state, the market structure is contingent on the state of the economy. Nevertheless, the strike is deterministic because it depends on the aggregate state at the purchase date and not at the payoff date. The case of constant strike for each option is a special case.

2.1.2 Idiosyncratic risk

In addition to the risky asset above, agents may invest in a productive technology. This investment is conditional on the arrival of investment opportunities, which may randomly vanish or appear in every period, and which are specific to every agent. We assume that this individual production (or opportunity) risk can neither be avoided nor insured and that it therefore constitutes an idiosyncratic risk, which agents must face on top of the previous aggregate risk. When the productive technology is available, agents are said to be *productive* (class p) and are able to freely invest in it. When it is not available, the so-called *unproductive* (class u) agents can only invest a fixed and suboptimal amount δ in the production technology. This production risk can be interpreted as a labor/employment risk or as an entrepreneurial risk.[5]

We assume that individual opportunity risk follows an independent first-order Markov chain, which is different for both types. When productive and being able to invest in the productive technology, type-i $(i = 1, 2)$ agents face a probability α^i of becoming unproductive, and thus a probability $1 - \alpha^i$ of remaining productive. When unproductive, type-i agents face a probability ρ^i of remaining unproductive. Transition probabilities being constant, the long-run fraction η^i of type-i productive agents equals: $\eta^i = \frac{1-\rho^i}{1+\alpha^i-\rho^i}$. The initial fraction of productive agents is also assumed to be η^i to avoid transitory dynamics.

Transition probabilities α^i and ρ^i are different among the two population types. These

[4] To avoid confusion, we devote the term *asset* to tree shares while *security* may designate either the option or the asset. *Derivative assets* can also be named *options*.

[5] Idiosyncratic risk is modeled in a similar way in Kiyotaki and Moore (**?**, **?**), Kocherlakota (**?**) and Miao and Wang (**?**).

different severities of the idiosyncratic risk is the sole source of ex-ante heterogeneity among agents. Ex post, agents of both types will differ according to the realization of their idiosyncratic shock. Derivative trading will result from the impact of this heterogeneity on the valuation of aggregate risk. Without loss of generality, we assume that type-1 productive agents face a higher probability of becoming unproductive and thus a higher individual risk: $\alpha^1 > \alpha^2$.

2.2 Agent's program

In each period, agents enjoy utility from consumption of a single good and suffer from the effort invested in the productive technology. Preferences are separable over time, in both consumption and effort and are represented by the instantaneous utility function $u(c) - e$, where the consumption is denoted c and the effort e. The function $u : \mathbb{R}^+ \to \mathbb{R}$ is twice differentiable, increasing and concave. We follow **?** and Lagos and Wright (**?**), in assuming that agents have a linear disutility of effort. Instantaneous preferences are discounted by a common factor of $\beta \in (0, 1)$, representing the time decay. We discuss below the implications of the linear disutility of effort.

The program of a type-i agent consists in maximizing his intertemporal utility under a set of constraints by choosing consumption, investment in the productive technology (if possible), asset and options demand, which are denoted respectively c_t^i, e_t^i, x_t^i and $s_t^{h,i}$ ($h = 1, \ldots, H$). The operator $E_0[\cdot]$ is the unconditional expectation over the aggregate and idiosyncratic shocks.

$$\max_{(c_t^i, e_t^i, x_t^i, (s_t^{i,h})_h)_{t \geq 0}} E_0 \left[\sum_{t=0}^{\infty} \beta^t \left(u(c_t^i) - e_t^i \right) \right] \tag{1}$$

$$\text{s.t.} c_t^i + P_t \, x_t^i + \sum_{h=1}^{H} Q_t^h s_t^{h,i} = \xi_t^i e_t^i + (1 - \xi_t^i)\delta + (P_t + y_t) \, x_{t-1}^i + \sum_{h=1}^{H} (P_t - K_{t-1}^h)^+ s_{t-1}^{h,i} \tag{2}$$

$$c_t^i \geq 0 \text{ and } e_t^i \geq 0 \tag{3}$$

$$P_t x_t^i + \sum_{h=1}^{H} Q_t^h s_t^{h,i} \geq 0, \tag{4}$$

$$x_t^i \geq 0, \tag{5}$$

$$\sum_{h=1}^{l} Q_t^h (s_t^{h,i,j})^+ \leq P_t \, x_t^{i,j}, \tag{6}$$

$$\lim_{t \to \infty} \beta^t E_0 \left[u'(c_t^i) \, x_t^i \right] = \lim_{t \to \infty} \beta^t E_0 \left[u'(c_t^i) \, s_t^{h,i} \right] = 0 \ (h = 1, \ldots, H), \tag{7}$$

$$\{x_{-1}^i, s_{-1}^{1,i}, \ldots, s_{-1}^{H,i}, \xi_0^i, y_0\} \text{ are given.} \tag{8}$$

Agents of type $i = 1, 2$ maximize their expected intertemporal utility (1), subject to a set of constraints (2)–(7) and initial security endowments and shocks (equation (8)). In constraint (2), total resources made up of production income (or inefficient production for the unproductive), asset dividends and asset-sale values are used to consume and to purchase assets. The second condition (3) states that both consumption and effort are positive, which will always be the case in all of the equilibria that we consider. The borrowing constraints stating that agents are prevented from holding a negative wealth appears in equation (4), while the one preventing them from selling the tree short is in equation (5). Condition (6) imposes a constraint stating that only the risky asset can be used as collateral. Indeed, with more than 2 options, an agent facing (4) and a binding constraint on asset holding ($x_t^{i,j} = 0$) could still use derivative assets for leverage. The agent could sell some options to buy other options, while respecting the constraint $\sum_{h=1}^{l} Q_t^h s_t^{h,i,j} = 0$. These portfolios are not relevant for our analysis because options are not used to hedge against the risky asset. Equation (7) sets out transversality conditions.

2.3 Equilibrium definition

We begin with expressing the security market clearing conditions. On one side, the security supply is equal to V for the asset and 0 for options. On the other side, we have to express aggregate individual security demands to compute the total security demand. To do so, we describe the distribution of type-i agents as a function of their security holdings and labor status using the probability measure $\Lambda_t^i : \mathcal{B}(\mathbb{R})^{1+H} \times \mathcal{B}(E^t) \to [0, 1]$.[6] This probability measure can be interpreted as follows: $\Lambda_t^i(X, S^1, \dots, S^H, I)$ (with $(X, S^1, \dots, S^H, I) \in \mathcal{B}(\mathbb{R})^{1+H} \times \mathcal{B}(E^t)$) is the measure of agents of type i, with asset holding $x \in X$, option positions $s^h \in S^h$ ($h = 1, \dots, H$), and with an individual history $\xi \in I$. Using the probability measures Λ^i, the market-clearing conditions become:

$$\sum_{i=1,2} \int_{\mathbb{R}^{1+H} \times E^t} x \Lambda_t^i(dx, ds^1, \dots, ds^H, d\xi) = V, \tag{9}$$

$$\sum_{i=1,2} \int_{\mathbb{R}^{1+H} \times E^t} s^h \Lambda_t^i(dx, ds^1, \dots, ds^H, d\xi) = 0 \ (h = 1, \dots, H). \tag{10}$$

The Walras law guarantees that the good market clears when the security markets clear. In this economy, a sequential competitive equilibrium can then be defined as follows:

Definition 1 (Sequential competitive equilibrium) *A sequential competitive equilibrium is a collection of consumption and effort levels $(c_t^i, e_t^i)_{t \geq 0}$, of asset demands $(x_t^i)_{t \geq 0}$, of derivative demands $(s_t^{1,i}, \dots, s_t^{H,i})_{t \geq 0}$ for $i = 1, 2$ and of security prices $(P_t, Q_t^1, \dots, Q_t^H)_{t \geq 0}$ such that for an initial distribution of security holdings, and of idiosyncratic and aggregate shocks $\{(x_{-1}^i, s_{-1}^{1,i} \dots, s_{-1}^{H,i}, \xi_0^i)_{i=1,2}, y_0\}$, we have:*

[6] For any metric space X, $\mathcal{B}(X)$ denotes the Borel sets of X.

1. *individual strategies solve the optimization program (1) when prices are given;*

2. *security prices adjust such that security markets clear at all dates and equations (9) and (10) hold;*

3. *the evolution of the probability measures Λ_t^1 and Λ_t^2 is consistent with individual choices.*

3 Reduced heterogeneity equilibrium

In heterogeneous-agent economies featuring both aggregate and uninsurable idiosyncratic shocks, it is usually not possible to explicitly derive the equilibrium. Indeed, it involves an infinite distribution of agents, who all have different individual histories. The usual strategy consists in computing approximate equilibria. In this paper, we derive an equilibrium where the heterogeneity in insurance demand can be computed with paper and pencil.

This equilibrium is based on two assumptions. The first, already introduced, is the linearity of the disutility in effort. When productive, agents freely adjust their effort to attain a constant marginal utility of consumption, equal to 1. All productive agents thus consume the same amount. This assumption enables to reduce the heterogeneity across productive agents. Our second assumption is that the supply V of the asset remains small enough such that even after selling off their entire portfolio, unproductive agents remain credit-constrained. The asset quantity is not sufficient for agents to overcome their credit constraint after becoming unproductive.

3.1 Equilibrium existence

The following proposition summarizes our existence result.

Proposition 1 (Equilibrium existence) *If:*

1. *the following condition holds:*

$$1 < u'(\delta) < 1 + \frac{1-\beta}{\beta\alpha^1}, \tag{11}$$

2. *the mass V of the tree is not too large,*

3. *the heterogeneity in idiosyncratic risk is limited (α^1 close to α^2),*

then there exists a limited-heterogeneity equilibrium characterized by equations (14)–(17).

We prove the existence of the limited-heterogeneity equilibrium in two steps: (i) we prove the result in an economy where both agent types face the same individual risk and where the asset supply is null; (ii) we extend the result by continuity to small asset supplies and not too different idiosyncratic risk. As far as we know, Miao (?) proves the sole general existence result in an economy featuring asset trades, credit constraints, and idiosyncratic and aggregate risk. We present here a "local" existence result, which allows for long-lived assets. Our equilibrium is a generalization of the equilibrium constructed in Challe, LeGrand and Ragot (?) to study the term structure of interest rates. Notice that we rule out sunspot equilibria: Contrary to ? or ?, options cannot play any role in our economy when the market is complete for the aggregate risk before the introduction of options.

Our limited-heterogeneity equilibrium exists under four conditions. Two of them are embedded in equation (11). The first one, $u'(\delta) < 1 + \frac{1-\beta}{\beta\alpha^1}$, ensures that asset prices are well-defined. If this condition does not hold, the price will be infinite because agents are too patient or their desire to self-insure is too high. This existence condition is less binding when the discount factor β is low or the idiosyncratic shock is not too severe. The second condition, $1 < u'(\delta)$, implies that unproductive agents are worse off compared to productive agents. The inefficient production level δ is thus not desirable. The third condition is that the asset volume remains small enough. Agents cannot hold too large an amount of assets to self-insure against idiosyncratic risk. The wealth constraint is binding for all unproductive agents. The fourth condition is that the heterogeneity remains limited, such that both agent types participate to both security markets. Intuitively, if agents face very different idiosyncratic risks, type-1 agents may be willing to pay a high price to self-insure against idiosyncratic risk. At such a high price, type-2 agents would like to short-sell the asset but, since they are prevented from doing so, options would not be traded.

3.2 Equilibrium characterization

Our equilibrium presents three particular features: (i) all unproductive agents are credit constrained and only productive agents trade assets; (ii) saving choices and asset demands of productive agents only depend on the current aggregate state and the agent's type; (iii) security prices only depend on the current aggregate state.[7] We therefore now use the subscript $k = 1, \ldots n$ instead of t for equilibrium variables. The equilibrium is characterized by a finite sequence of $3(1+H)n$ variables $(x_k^i, s_k^{1,i}, \ldots, s_k^{H,i}, P_k, Q_k^1, \ldots, Q_k^H)_{k=1,\ldots,n}^{i=1,2}$ instead of continuous distributions as in standard incomplete market models.

We now characterize the equilibrium by construction. Productive agents equalize the marginal utility of consumption to the marginal pain of effort equal to 1. As a consequence, the consumption of any productive type-i agent, $c^{p,i}$, satisfies $u'(c^{p,i}) = 1$.[8] Second, budget

[7] Such equilibrium features, particularly that unproductive agents do not trade, are also present in Kiyotaki and Moore (?, ?), and Miao and Wang (?).

[8] The marginal utility of productive agents is constant and independent of the consumption level. The

constraints provide the consumption and the effort of every agent as a function of the current and past aggregate and idiosyncratic shocks. For instance, a type-i agent who was productive in $t-1$ in state k, and is unproductive in t in state j, consumes $c_{kj}^{pu,i}$ equal to the production δ plus the liquidation value of his portfolio:

$$c_{kj}^{pu,i} = \delta + (P_j + y_j)\, x_k^1 + \sum_{h=1}^{H} (P_j - K_k^h)^+ s_k^{h,1}. \tag{12}$$

By the same token, the consumption and effort of any agent can be shown to depend on his type and on the current and past aggregate and idiosyncratic shocks. The number of consumption levels is then finite (16 at most).

We now express the pricing kernel $M_{t,t+1}^i = \beta u'(c_{t+1}^i)/u'(c_t^i)$ of agents participating in financial markets, which determines equilibrium security prices. A type-i productive agent at t in state k, has a marginal utility of consumption equal to 1. His next period marginal utility depends on his productive status in period $t+1$. If he stays productive (with probability $1-\alpha^i$), his marginal utility will be 1. If he becomes unproductive, his consumption will be given by equation (12). Let j be the aggregate state at date $t+1$. Then the pricing kernel M_{kj}^i of this type i agent is:

$$M_{kj}^i = \beta \pi_{k,j} \left(1 + \alpha^i \left(u' \left(\delta + (P_j + y_j) x_k^i + \sum_{h=1}^{H} (P_j - K_k^h)^+ s_k^{h,i} \right) - 1 \right) \right). \tag{13}$$

Two conditions ensure that unproductive agents of both types do not hold securities. First, $P_t u'(c_t^i) > \beta E_t[u'(c_{t+1}^i)(P_{t+1} + y_{t+1})]$ ensures that they do not hold the asset while $Q_t^h u'(c_t^i) > \beta E_t[u'(c_{t+1}^i)(P_{t+1} - K_t^h)^+]$ (for any h) ensures that they will not hold any option. These conditions simply mean that security prices are too high for unproductive agents. They can be expressed as a function of the variables characterizing the equilibrium $(x_k^i, s_k^{1,i}, \ldots, s_k^{H,i}, P_k, Q_k^1, \ldots, Q_k^H)_{k=1,\ldots,n}^{i=1,2}$ and are shown to hold in equilibrium.

We summarize the full equilibrium structure in the next proposition.

Proposition 2 (Equilibrium properties) *The reduced-heterogeneity equilibrium characterized by the set of quantities and prices* $(x_k^i, s_k^i, P_k, Q_k)_{k=B,G}^{i=1,2}$ *solves the following equa-*

irrelevance result of Krueger and Lustig (**?**) therefore cannot apply in our setup. Indeed, as explained in the introduction, their result relies on the assumption that all marginal utilities are homogeneous with the same degree, which implies aggregate shocks have the same impact on asset prices no matter the degree of market incompleteness.

tions ($k = 1, \ldots, n,\ h = 1, \ldots, H$ and $i = 1, 2$):

$$P_k = \beta \sum_{j=1}^{n} \pi_{k,j} (1 + \alpha^i (u'(\delta + (P_j + y_j)\, x_k^i + \sum_{h=1}^{H} (P_j - K_k^h)^+ s_k^{h,i}) - 1))(P_j + y_j) \tag{14}$$

$$Q_k^h = \beta \sum_{j=1}^{n} \pi_{k,j} (1 + \alpha^i (u'(\delta + (P_j + y_j)\, x_k^i + \sum_{l=1}^{H} (P_j - K_k^l)^+ s_k^{l,i}) - 1))(P_j - K_k^h)^+ \tag{15}$$

$$V = \eta^1\, x_k^1 + \eta^2\, x_k^2, \tag{16}$$

$$0 = \eta^1\, s_k^{h,1} + \eta^2\, s_k^{h,2}. \tag{17}$$

The equilibrium is characterized by four sets of equations (which amount to $3(1+H)n$ equations). The first set, (14), concerns the price of the asset. It equates the asset price in each state of the world to the expectation of the asset payoff discounted by the pricing kernel, (13). Equations (15) have a similar interpretation, equalizing the option price to the expected option payoff discounted by the pricing kernel. Finally, the last two sets of equations, (16) and (17), are the market clearing conditions for the asset and the H options in the n states of the world. Equilibrium quantities must also verify the set of inequalities (28) and (29) in Appendix that guarantee that unproductive agents do not trade any financial asset.

Moreover, effort quantities of productive agents at the equilibrium are strictly interior. First, they cannot be infinite, or it would otherwise imply that budget constraints are not binding, which would be dominated by consuming up to the budget constraints. Second, effort quantities are positive at the equilibrium, since they are determined by the budget constraint (2) in which security holdings are small.

These equations can easily be simulated. More importantly, this equilibrium allows us to derive theoretical insights about the exchange of insurance against aggregate risk throughout the business cycle. Our main theoretical findings are provided in Section 4.

4 Interactions between aggregate uncertainty and heterogeneity

In this section, we make additional assumptions to derive theoretical results about the effect of aggregate uncertainty and heterogeneity on risk sharing. These results characterize the price and volume of derivative assets in the business cycle.

4.1 The simple economy

We consider an economy with two aggregate states and a single option: $n = 2$ and $H = 1$. We denote by a subscript G (B) the good (bad) state: $y_G > y_B$. Conditions of Proposition

1 are supposed to hold and an equilibrium exists. We make three additional assumptions.

1. *Aggregate states are persistent, i.e.* $\pi_{GG} + \pi_{BB} > 1$.
   To avoid the discussion of uninteresting cases, we assume that all eigenvalues of the transition matrix for the aggregate risk are positive; this implies that aggregate states are persistent and do not fluctuate too frequently.[9]

2. *The utility function u is such that:*

$$X \mapsto -X \frac{u''(\delta + X)}{u'(\delta + X) - 1} \text{ is increasing for } X \in [0, u'^{-1}(1) - \delta). \tag{18}$$

   This assumption is quite general as it always holds for standard utility classes such as CRRA, CARA, and quadratic utilities. This condition determines the direction of risk sharing. Indeed, it describes how the difference in marginal utilities between being non-productive and being productive (i.e. $u'(\delta + X) - 1$, which can be interpreted as the extensive cost of losing a production opportunity) vary with better insurance. Notably, it implies that the extensive cost of losing a production opportunity diminishes when security payoffs increase.

3. *The strike K of the option is such that the option exactly pays off in one state of the world.*
   This condition guarantees that the option is not a trivial asset. There are two other cases, which are uninteresting. (i) The strike is always greater than the asset price. The option then never pays off and cannot be considered as an actual asset. (ii) The strike is always smaller than the asset price. The option then pays off in both states of the world and is thus redundant with the asset.
   This assumption is obviously stated as an equilibrium requirement but corresponds to implicit conditions on deep model parameters. Moreover, the strike can be constructed as follows. In the zero supply economy ($V = 0$), the option cannot be traded and the option introduction has no impact on asset prices. It is straightforward to pick up a proper strike. Since an increase in the asset supply continuously modifies asset prices, it remains possible, when asset volume is positive, to choose a strike for the option to pay off in exactly one state of the world.
   With our assumption (and in particular the persistence of aggregate states), the asset price in the good state can be shown to be larger than in the bad one. The option therefore only pays off in the good state of the world.

Following the above assumption, the two states of the world correspond to two non-redundant securities (the asset and the option). Markets are thus complete for the aggregate risk. Equations (14) to (17) then imply that every agent type holds a security

[9] For instance, Hamilton (1994, chapter 22) finds $\pi_{GG} + \pi_{BB} = 1.65$ for quarterly US data.

portfolio, which is independent of the state of the world. We can therefore simplify the equilibrium defined by equations (14) to (17). The equilibrium is characterized by eight variables $\{x^1, x^2, s^1, s^2, P_G, P_B, Q_G, Q_B\}$ together with the following eight equations:

$$\alpha^1 \left(u'(\delta + (P_B + y_B)x^1) - 1\right) = \alpha^2 \left(u'(\delta + (P_B + y_B)x^2) - 1\right) \tag{19}$$

$$\alpha^1 \left(u'(\delta + (P_G + y_G)x^1 + (P_G - K)s^1) - 1\right) = \alpha^2 \left(u'(\delta + (P_G + y_G)x^2 + (P_G - K)s^2) - 1\right) \tag{20}$$

$$\begin{aligned} P_k = {} & \beta\pi_{k,G} \left(1 + \alpha_k^1 \left(u'\left(\delta + (P_G + y_G)x^1 + (P_G - K)s^1\right) - 1\right)\right)(P_G + y_G) \\ & + \beta\pi_{k,B} \left(1 + \alpha_k^1 \left(u'\left(\delta + (P_B + y_B)x^1\right) - 1\right)\right)(P_B + y_B), \;\; k = G, B \end{aligned} \tag{21}$$

$$Q_k = \beta\pi_{k,G} \left(1 + \alpha_k^1 \left(u'\left(\delta + (P_G + y_G)x^1 + (P_G - K)s^1\right) - 1\right)\right)(P_G - K), \;\; k = G, B \tag{22}$$

$$V = \eta^1 x^1 + \eta^2 x^2 \tag{23}$$

$$0 = \eta^1 s^1 + \eta^2 s^2 \tag{24}$$

Equations (19) and (20) stem from the market completeness for aggregate risk. These equations guarantee that both agent types participate in both financial markets and that they manage to equalize their expected marginal utility (expectation with respect to individual risk) across both states of the world. Equations (21) and (22) are the pricing equations for the asset and the option in each state of the world. The pricing is made only by type-1 agents but, thanks to (19) and (20), it is the same for type-2 agents. Finally, equations (23) and (24) are two market clearing conditions. Propositions 3–5 gather our main results derived from the analysis of this equilibrium.

4.2 Portfolio compositions

The next proposition characterizes equilibrium portfolios.

Proposition 3 (Agents' portfolios) *Type-*1 *agents, facing a greater risk of becoming unproductive, choose to hold a greater quantity of assets than type-2 agents, i.e.* $x^1 > x^2 > 0$*. Further, type-1 agents sell options to type-2:* $s^1 < 0 < s^2$*.*

Since options only pay off in the good state of the world and not in the bad, both agent types choose the quantity of assets, x^1 and x^2, so as to equalize their marginal utility in the bad sate of the world (equation (19)). Type-1 agents, who are more likely to become unproductive, purchase a greater quantity of assets to self-insure against the risk of losing the production opportunity in the bad state. Option holdings are determined by the equalization of expected marginal utilities in the good state of the world (equation (20)). In the good state, the asset is a better insurance device than in the bad state (because of its higher sale price and higher dividend), which benefits type-1 agents more (since they hold more assets and since condition (18) holds). As a consequence, type-1 agents sell options in order to reduce the liquidation value of their portfolio in the good state of the world. On the other hand, type-2 agents purchase options to improve their hedging in the good state of the world.

From a financial point of view, type-1 agents choose a kind of "delta-hedging" strategy in equilibrium, in the sense that they optimally hold a portfolio that is less affected by variations in the underlying asset price than it would be without options.

These results are not straightforward to compare with the ones of **?**. Indeed, the latter consider an alternative representation of uninsurable risk and a complete set of derivative assets depending on the aggregate payoff of the economy (and not, as we do, derivative assets on the price of the underlying asset). **?** (**?**, Theorem 3) find that agents with relatively high idiosyncratic risk tend to buy claims with convex payoffs and those with relatively low idiosyncratic risk tend to sell those claims. Although they manage to nicely characterize individual sharing rules, they do not study the implications of idiosyncratic risk on the volume of traded options. Conversely, we specify the market structure of derivative assets to be able to characterize option volumes (and thus the exchange of insurance on the aggregate risk) along the business cycle. However, note that our results could be consistent with **?**'s, if we introduce further heterogeneity. For instance, assume that type-1 and type-2 agents have different suboptimal production levels δ^1 and δ^2. We can then prove that type-1 agents can buy call options to type-2 agents, even if the former have a higher probability α^1 to be unproductive (i.e., a higher probability to face a bad idiosyncratic outcome) than the latter. Lemma 2 in Section F of the Appendix formally states and proves this result. Even if this is possible in our setup, our initial modeling choice is supported by many empirical studies using micro-level data on both households and institutional investors. In particular, investors facing a severe individual risk are mainly found to hold less risky assets or to participate less in financial markets.[10]

One may also wonder what would be the consequences of the introduction of a riskless asset, which seems to be a natural extension of our setup. In Appendix G, we prove that our results still hold and are compatible with the fact that high-risk agents hold less risky portfolios than low-risk agents. In a three-state economy with three non-redundant securities (bonds, stocks and calls), high-risk agents purchase more bonds than low-risk agents, and always sell calls to hedge their stock portfolios. High-risk agents therefore hold more riskless assets and better hedged portfolios of stocks and options. Proposition G.3 gathers these findings. This result is consistent with Krusell and Smith (**?**), who numerically show that in an economy with only riskless bonds and risky stocks, low-wealth households hold only bonds and high-wealth households hold all the stocks.

We now explain how aggregate uncertainty may affect security holdings and prices. This is the main theoretical result concerning the dynamic behavior of option volumes.

[10]For example, see Haliassos and Bertaut (**?**) for a study on U.S. households and Guiso, Jappelli and Terlizzese (**?**) for one on European households.

Proposition 4 (Aggregate uncertainty effects) *An increase in aggregate uncertainty through a mean-preserving spread in dividends in the vicinity of the riskless equilibrium (i.e. $y_G = y_B$) has the following consequences:*

1. *The asset price rises in the good state and falls in the bad state, while the average price remains unchanged;*

2. *The option price increases in both states, but more in the good than in the bad state;*

3. *The higher the volatility of dividends, the greater the volume of options traded.*

We consider a mean-preserving spread in dividends, which increases aggregate uncertainty while remaining close to the riskless equilibrium. While the asset payoff in the bad state of the world, $P_B + y_B$, decreases, the asset payoff in the good state, $P_G + y_G$, increases. Due to the aggregate-state persistence, the spread between asset prices in both states therefore increases with the mean-preserving spread even though the average price does not change (result 1). The option payoff, $P_G - K$, increases in the good state and still pays off 0 in the bad one. As a consequence, the price of the option increases in both states. Moreover, since aggregate states are persistent, this increase is greater in the good state of the world (result 2).

This mean-preserving spread in dividends also has an impact on portfolio compositions. Due to a smaller dividend in the bad state, type-1 agents, who are more likely to be unproductive, purchase a greater quantity of assets to obtain sufficient hedging in the bad state of the world to equalize marginal utility with type-2 agents (equation (19)). They consequently sell a greater quantity of options in order to reduce the portfolio liquidation value in the good state of the world (equation (20)), while type-2 agents purchase these options in order to increase their hedging in the good state of the world, causing the quantity of traded options to increase (result 3). Our model is therefore able to reproduce the stylized fact that the open interest of options, i.e. the number of open contracts, which is similar to $|s^1|$ in our economy, rises as asset prices become more volatile. This stylized fact has been empirically highlighted for options on the S&P 500 by **?** or **?**.

The degree of market incompleteness also has consequences for portfolio composition. These results are given in the next proposition:

Proposition 5 (Heterogeneity effects) *An increase in the production opportunity risk for type-1 agents (i.e. a higher α^1) in the vicinity of the symmetric equilibrium (i.e. $\alpha^1 = \alpha^2$) has the following consequences:*

1. *The prices of both securities increase in both states of the world, but more in the good state than in the bad;*

2. *Type*-1 *agents purchase more assets and sell more options.*

Greater heterogeneity therefore increases the volume of traded options.

We now consider an increase in the opportunity risk for type-1 agents (i.e. a higher α^1) in the vicinity of the symmetric equilibrium. Type-1 agents experience a greater probability of becoming unproductive. They therefore express a greater demand for self-insurance, resulting in higher prices for both securities. The price increase is greater in the good state of the world, when both securities are a better hedge against having no production opportunities, due to aggregate-state persistence (result 1). The increase in demand for insurance by type-1 agents translates into a greater demand for the asset for insurance in the bad state of the world (equation (19)) and into a greater quantity of options sold for insurance in the good state of the world (equation (20)), which proves the second point of Proposition 5.

Another avenue to understanding the results of Proposition 5 consists in directly interpreting the probability α^1 as a parameter driving the strength of the demand for securities. Indeed, α^1 drives the severity of the idiosyncratic risk and thus the demand for the self-insurance. As stated in Proposition 5, an increase in the demand pressure through a larger α^1 means, first, a greater option trading volume and a higher option price, as well as a higher asset price. This demand-pressure channel on option prices has been empirically highlighted, for instance by Bollen and Walley (**?**). Our model therefore provides a possible micro-foundation for this demand-pressure effect in a general equilibrium setting.

5 Conclusion

The goal of this paper is to contribute to the theoretical understanding of incomplete market models with aggregate risk. Using a framework that endogenously generates a limited heterogeneity equilibrium, we show that these environments are useful for understanding both allocations and prices of derivative assets. In particular, we show that these environments generate more option trades in bad times than in good times. A first extension of this work could be to quantitatively investigate the properties of a calibrated model. A second and possibly more promising application could be to use this simple model to estimate the degree of heterogeneity from the actual data on the volume of traded options. We leave these quantitative investigations for future research.

Appendix

A Proof of Proposition 1

We proceed in two steps: (i) we prove that our equilibrium exists in an economy without heterogeneity and with a zero mass tree and (ii) we check that the result still holds by continuity.

First step: Equilibrium existence with zero volume and no heterogeneity. We first assume $\alpha^1 = \alpha^2$ and $V = 0$. The asset price is given by:

$$P_k = \beta \left(1 + \alpha^1 \left(u'(\delta) - 1\right)\right) \sum_{j=1}^{n} \pi_{kj} \left[P_j + y_j\right]. \tag{25}$$

The first part of (11) stating that $\beta \left(1 + \alpha^1 \left(u'(\delta) - 1\right)\right) < 1$ guarantees that the price P_k is well defined in any state k. The condition of non-participation of unproductive agents is: $P_k u'(\delta) > \beta(1 + \rho(u'(\delta) - 1)) \sum_{j=1}^{n} \pi_{kj}(P_j + y_j)$. Indeed, the collateral constraint (6) implies that agents cannot short-sell some options to purchase other options: in absence of assets, options cannot be traded. Using (25), the condition becomes: $\left(1 + \alpha^1 \left(u'(\delta) - 1\right)\right) u'(\delta) > u'(\delta) - 1 > 1 + \rho \left(u'(\delta) - 1\right)$, which always holds whenever $u'(\delta) > 1$. No trade occurs.

Second step: Positive supply economy with one type of agent. We assume $\alpha^2 = \alpha^1$ but $V > 0$. Every (identical) productive agent holds the same asset quantity $\frac{V}{\eta^1 + \eta^2}$. Options are still not traded in this economy. The asset price P_k in state $k = 1, \ldots, n$ verifies the following equation:

$$P_k = \beta \sum_{j=1}^{n} \pi_{kj} \left(1 + \alpha^1 \left(u'(\delta + (P_j + y_j) \frac{V}{\eta^1 + \eta^2}) - 1\right)\right) (P_j + y_j). \tag{26}$$

The condition for equilibrium existence is:

$$P_k u'(\delta + (P_k + y_k) \frac{V}{\eta^1 + \eta^2}) > \beta \sum_{k'=1}^{n} \pi_{kk'} \left(1 + \rho \left(u'(\delta) - 1\right)\right) (P_{k'} + y_{k'}). \tag{27}$$

We can express (26) as $G(P, V) = 0$, where $P = (P_k)_{k=1,\ldots,n} \in (\mathbb{R}^+)^n$ and $G : (\mathbb{R}^+)^n \times \mathbb{R}^+ \to (\mathbb{R}^+)^n$ is continuous and differentiable in V. The Jacobian relative to P in $V = 0$ is $G_P(P, 0) = (1_{k=j} - \beta \pi_{kj} (1 + \alpha(u'(\delta) - 1)))_{k,j=1,\ldots,n} > 0$.[11] This Jacobian matrix can be written as $I_n - \widetilde{G}$, where I_n is the identity matrix and $\widetilde{G}$ a matrix whose norm is strictly smaller than one. The Jacobian G_P is invertible and the implicit-function theorem implies that (26) defines P as a continuous function of V around $V = 0$. As condition (27) holds for $V = 0$, there exists a neighborhood $W_1(0) \subset \mathbb{R}^+$ where (27) holds. We define $V^* \in W_1(0) > 0$ and P^* (Q^*) the corresponding asset (option) price. The quantity of assets held by each agent is $x^* = \frac{V^*}{\eta^1 + \eta^2} > 0$, while no option is traded.

Third step: Positive supply economy with two types of agents. In the general case, equilibrium quantities $(P_k, Q_k^1, \ldots, Q_k^l, x_k^i, s_k^{1,i}, \ldots, s_k^{l,i})_{k=1,\ldots,n}^{i=1,2}$ are characterized by equations (14)–(17) and must verify the following inequalities ($k, \hat{k} = 1, \ldots, n$; $h = 1, \ldots, l$; $i = 1, 2$):

$$P_k u'(\delta + (P_k + y_k)\, x_{\hat{k}}^i + \sum_{l=1}^{H} (P_k - K_{\hat{k}}^l)^+ s_{\hat{k}}^{l,i}) > \beta \sum_{j=1}^{n} \pi_{kj} \left(1 + \rho \left(u'(\delta) - 1\right)\right) (P_j + y_j), \tag{28}$$

$$Q_k^h u'(\delta + (P_k + y_k)\, x_{\hat{k}}^i + \sum_{l=1}^{H} (P_k - K_{\hat{k}}^l)^+ s_{\hat{k}}^{l,i}) > \beta \sum_{j=1}^{n} \pi_{kk'} \left(1 + \rho \left(u'(\delta) - 1\right)\right) (P_j + y_j). \tag{29}$$

[11]The function $1_{k=j}$ equals 1 when $k = j$ and 0 otherwise.

We proceed in a similar vein as above. We define $X = ((y_k)_k, V, (\alpha_k^1, \alpha_k^2)_k) \in (\mathbb{R}^+)^{n\times 1\times 2n}$ as the vector of parameters and $Z = ((P_k)_k, (Q_k^h)_k^h, (x_k^1)_k, (x_k^2)_k, (s_k^{1h})_k^h, (x_k^{2h})_k^h) \in (\mathbb{R}^+)^{n\times nl\times 2n\times 2nl}$ as the vector of endogenous variables. We define the function F stacking pricing functions for both agent types and the market equilibrium equations (i.e., equations (14)–(17)) such that for a given set of parameters X, an equilibrium Z is defined as a solution of $F(Z, X) = 0$.

From the previous step, we know that there exists an equilibrium for $X^* = ((y_k)_k, V^*)$ in which the unproductive do not trade assets; this equilibrium is defined by $Z^* = (P^*, Q^*, (x^*, \ldots, x^*), 0_{2nl})$. We now show that the Jacobian $\Delta = \left(\frac{\partial F_i}{\partial z_j}(Z^*, X^*)\right)_{i,j=1,\ldots,3n(1+l)}$ is invertible.

In the vicinity of the symmetric equilibrium, Δ has the following shape:

$$\Delta = \begin{bmatrix} I_n - A & 0_{n\times nl} & K_a & 0_{n\times n} & (K_{a1}\ldots K_{al}) & 0_{n\times nl} \\ I_n - A & 0_{n\times nl} & 0_{n\times n} & K_a & 0_{n\times nl} & (K_{a1}\ldots K_{al}) \\ -\begin{pmatrix} B_1 \\ \vdots \\ B_l \end{pmatrix} & I_{nl} & \begin{pmatrix} K_{a1} \\ \vdots \\ K_{al} \end{pmatrix} & 0_{nl\times n} & (K_{gh})_{g,h=1,\ldots,l} & 0_{nl\times n} \\ -\begin{pmatrix} B_1 \\ \vdots \\ B_l \end{pmatrix} & I_{nl} & 0_{nl\times n} & \begin{pmatrix} K_{a1} \\ \vdots \\ K_{al} \end{pmatrix} & 0_{nl\times n} & (K_{gh})_{g,h=1,\ldots,l} \\ 0_{n\times n} & 0_{n\times nl} & E_1 & E_2 & 0_{n\times nl} & 0_{n\times nl} \\ 0_{nl\times n} & 0_{nl\times nl} & 0_{nl\times n} & 0_{nl\times n} & E_1 \otimes I_l & E_2 \otimes I_l \end{bmatrix},$$

where:

- I_p is the $p \times p$ identity matrix, $0_{n\times p}$ is the $n \times p$ null matrix, $1_{l\times 1}$ is a column vector of length l containing only 1 and $\otimes$ is the Kronecker product;
- A is an $n \times n$ matrix such that $A_{k,j} = \beta\pi_{k,j}(1 + \alpha^1(u'(\delta + (P_j^* + y_j)x^*) - 1 + x^* u''(\delta + (P_j^* + y_j)x^*)(P_j^* + y_j)))$;
- B_h $(h = 1, \ldots, l)$ is an $n \times n$ matrix such that $B_{h,kj} = \beta\pi_{k,j}(1 + \alpha_k^1((u'(\delta + (P_j^* + y_j)x^*) - 1)1_{P_j \geq K_k^h} + x^* u''(\delta + (P_j^* + y_j)x^*)(P_j - K_k^h)^+))$;
- E_i $(i = 1, 2)$ is an $n \times n$ diagonal matrix such that $E_{i,kk} = \eta_k^i$;
- K_a is an $n \times n$ diagonal matrix such that $K_{a,kk} = -\beta\alpha_k^1 \sum_{j=1}^n \pi_{k,j} u''(\delta + (P_j^* + y_j)x^*)(P_j^* + y_j)^2$;
- K_{ah} is an $n \times n$ diagonal matrix such that $K_{ah,kk} = -\beta\alpha_k^1 \sum_{j=1}^n \pi_{k,j} u''(\delta + (P_j^* + y_j)x^*)(P_j^* + y_j)(P_j - K_k^h)^+$;
- K_{gh} is an $n \times n$ diagonal matrix such that $K_{gh,kk} = -\beta\alpha_k^1 \sum_{j=1}^n \pi_{k,j} u''(\delta + (P_j^* + y_j)x^*)(P_j^* - K_k^g)^+(P_j^* - K_k^h)^+$.

We now prove that Δ is invertible. Let $X = (X_1, \ldots, X_6) \in (\mathbb{R})^{n+nl+2n+2nl}$. $X \in \ker \Delta$ implies the following set of equalities:[12]

$$\begin{cases} 0_{n\times 1} = & (I_n - A)X_1 + K_a X_3 + (K_{a1}\ldots K_{al})X_5 \\ 0_{n\times 1} = & (I_n - A)X_1 + K_a X_4 + (K_{a1}\ldots K_{al})X_6 \\ 0_{nl\times 1} = & -(B_1\ldots B_l)^\top X_1 + X_2 + (K_{a1}\ldots K_{al})^\top X_3 + (K_{gh})X_5 \\ 0_{nl\times 1} = & -(B_1\ldots B_l)^\top X_1 + X_2 + (K_{a1}\ldots K_{al})^\top X_4 + (K_{gh})X_6 \\ 0_{n\times 1} = & E_1 X_3 + E_2 X_4 \\ 0_{nl\times 1} = & E_1 \otimes I_l X_5 + E_2 \otimes I_l X_6 \end{cases}. \tag{30}$$

[12] $M^\top$ denotes the transpose of the matrix M.

Using the two first equations with the two last ones (together with the fact that any two diagonal matrices commute), we obtain $X_1 = 0$. By the same token, using the second and third equations with the two last ones, we obtain $X_2 = 0$. The system (30) simplifies to:

$$\begin{cases} 0_{n\times 1} = & K_a X_3 + (K_{a1}\dots K_{al})X_5 \\ 0_{n\times 1} = & K_a X_4 + (K_{a1}\dots K_{al})X_6 \\ 0_{nl\times 1} = & (B_1\dots B_l)^\top X_3 + (K_{gh})X_5 \\ 0_{nl\times 1} = & (B_1\dots B_l)^\top X_4 + (K_{gh})X_6 \\ 0_{n\times 1} = & E_1 X_3 + E_2 X_4 \\ 0_{nl\times 1} = & E_1 \otimes I_l X_5 + E_2 \otimes I_l X_6 \end{cases}. \tag{31}$$

Since K_a is invertible, we obtain $0_{nl\times 1} = (\begin{pmatrix} K_{a1} \\ \vdots \\ K_{al} \end{pmatrix} (K_a^{-1}K_{a1}\dots K_a^{-1}K_{al}) - (K_{gh}))X_5$, which implies $X_5 = 0_{nl\times 1}$. To see this, we express $X_5 = (X_{51}, \dots, X_{5l}) \in (\mathbb{R}^n)^l$ and get that for any $g = 1, \dots, l$, we have:

$$\sum_{h=1}^{l} \left(K_{ag}K_{ah} - K_a K_{gh}\right) X_{5h} = 0. \tag{32}$$

We introduce the bilinear form $(\cdot|\cdot)_k$: $\forall U, V \in (\mathbb{R}^n)^2$, $(U|V)_k = -\beta\alpha^1 \sum_{j=1}^n \pi_{k,j} u''_{k,j} U_j V_j$ and it is easy to check that it is an inner product. Multiplying (32) by X_{5g} and summing for $g = 1, \dots, l$, we obtain using $(\cdot|\cdot)_k$: $(\sum_{h=1}^l \Pi_h^k X_{5hk}|\Pi_a)_k^2 - (\Pi_a|\Pi_a)_k(\sum_{h=1}^l \Pi_h^k X_{5hk}|\sum_{h=1}^l \Pi_h^k X_{5hk})_k = 0$, where $\Pi_a = (P_j^* + y_j)_{j=1,\dots,n}$ and $\Pi_h^h = (P_j^* - K_k^h)^+_{j=1,\dots,n}$. The Cauchy-Schwarz inequality implies then that for all k, $\lambda_k \Pi_a = \sum_{h=1}^l \Pi_h^k X_{5hk}$, which means that $\lambda_k = 0$ and $X_{5h} = 0_{n\times 1}$, since by assumption option payoffs do not replicate asset payoffs. From the first equation in (31), we deduce $X_3 = 0_{n\times 1}$. By the same token, we obtain $X_6 = 0_{nl\times 1}$ and $X_4 = 0_{n\times 1}$.

We conclude that $\ker\Delta = \{0\}$ and the Jacobian Δ in (X^*, Z^*) is invertible. The implicit-function theorem proves that there exists a continuously differentiable function $\widetilde{F}$ such that $Z = \widetilde{F}(X)$ for X close to X^*. In consequence, our equilibrium exists in the vicinity of (X^*, Z^*).

B Proof of Proposition 2

From Section 3.2, it is straightforward to deduce the price expressions (14) and (15). Market clearing conditions (16) and (17) are easily deduced from equilibrium properties (only productive agents trade securities) and from the general market clearing conditions (9) and (10).

As explained in Section 3.2, two conditions have to hold for preventing unproductive agents to trade any securities. These conditions are $P_t u'(c_t^i) > \beta E_t[u'(c_{t+1}^i)(P_{t+1} + y_{t+1})]$ and $Q_t^h u'(c_t^i) > \beta E_t[u'(c_{t+1}^i)(P_{t+1} - K_t^h)^+]$ for all unproductive agents. Plugging price expressions, we obtain the equations (28) and (29) above.

C Proof of Proposition 3

Since $\alpha^1 > \alpha^2$, (19) implies that $u'\left(\delta + (P_B + y_B)\, x^1\right) < u'\left(\delta + (P_B + y_B)\, x^2\right)$ and therefore that $x^1 > x^2$, since u' is decreasing. Moreover, $s^1 < 0$ if $\alpha^1(u'\left(\delta + (P_G + y_G)\, x^1\right) - 1) < \alpha^2(u'\left(\delta + (P_G + y_G)\, x^2\right) - 1)$ or if $\frac{u'\left(\delta+(P_G+y_G)\,x^1\right)-1}{u'\left(\delta+(P_G+y_G)\,x^2\right)-1} < \frac{u'\left(\delta+(P_B+y_B)\,x^1\right)-1}{u'\left(\delta+(P_B+y_B)\,x^2\right)-1}$. This holds if $\pi \mapsto \frac{u'\left(\delta+\pi x^1\right)-1}{u'\left(\delta+\pi x^2\right)-1}$ is decreasing, which is guaranteed by condition 18.

D Proof of Proposition 4

We introduce as a benchmark the equilibrium without aggregate risk in which $y_G = y_B = y$. The asset (option) price is P^f (Q^f). Options are not traded ($s^{f,i} = 0$) and the asset holdings of type-i agents are denoted $x^{f,i}$, $i = 1, 2$:

$$\frac{P^f}{P^f + y} = \beta\left(1 + \alpha^i\left(u'\left(\delta + (P^f + y)x^{f,i}\right) - 1\right)\right) = \frac{Q^f}{P^f - K}, \tag{33}$$

$$\text{with: } \alpha^1\left(u'(\delta + (P^f + y)\,x^{f,1}) - 1\right) = \alpha^2\left(u'(\delta + (P^f + y)\,x^{f,2}) - 1\right)\ ;\ \eta^1 x^{f,1} + \eta^2 x^{f,2} = V. \tag{34}$$

Since $\alpha^1 > \alpha^2$, we deduce that $x^{f,1} > x^{f,2}$; in the absence of aggregate risk, the agent facing the larger risk holds more assets. We also introduce the following constant κ:

$$\kappa = \frac{-x^{f,1}\frac{u''\left(\delta+(P^f+y)\,x^{f,1}\right)}{u'(\delta+(P^f+y)\,x^{f,1})-1} + x^{f,2}\frac{u''\left(\delta+(P^f+y)\,x^{f,2}\right)}{u'(\delta+(P^f+y)\,x^{f,2})-1}}{-\frac{1}{\eta^1}\frac{u''(\delta+(P^f+y)\,x^{f,1})}{u'(\delta+(P^f+y)\,x^{f,1})-1} - \frac{1}{\eta^2}\frac{u''(\delta+(P^f+y)\,x^{f,2})}{u'(\delta+(P^f+y)\,x^{f,2})-1}} > 0. \tag{35}$$

Condition 18 guarantees that $\kappa > 0$. Before going further, we prove the following lemma:

Lemma 1 *Let Φ be a real continuously differentiable function of y_G and y_B. We denote by $V[y]$ ($E[y]$) the variance (mean) of the process y. A mean-preserving spread of y implies:*

$$\left.\frac{\partial \Phi}{\partial V[y]}\right|_{E[y]=constant} = \frac{1}{2(y_G - y_B)}\left[\frac{2 - \pi_{GG} - \pi_{BB}}{1 - \pi_{BB}}\frac{\partial \Phi}{\partial y_G} - \frac{2 - \pi_{GG} - \pi_{BB}}{1 - \pi_{GG}}\frac{\partial \Phi}{\partial y_B}\right]. \tag{36}$$

Proof:
Defining $q = \frac{1-\pi_{BB}}{2-\pi_{GG}-\pi_{BB}}$, we have $y_G = E[y] + (1-q)\sqrt{\frac{V[y]}{q(1-q)}}$ and $y_B = E[y] - q\sqrt{\frac{V[y]}{q(1-q)}}$. Using basic differential calculus, it is straightforward to derive (36).

D.1 Quantities

Deriving (19) and (20) relative to y_l in the vicinity of the riskless equilibrium yields:

$$\alpha^1 u''\left(\delta + (P^f + y)\,x^{f,1}\right)\left(x^{f,1}(\frac{\partial P_B}{\partial y_l} + 1_{l=B}) + (P^f + y)\frac{\partial x^1}{\partial y_l}\right)$$
$$= \alpha^2 u''\left(\delta + (P^f + y)\,x^{f,2}\right)\left(x^{f,2}(\frac{\partial P_B}{\partial y_l} + 1_{l=B}) + (P^f + y)\frac{\partial x^2}{\partial y_l}\right),$$
$$\alpha^1 u''\left(\delta + (P^f + y)\,x^{f,1}\right)\left(x^{f,1}(\frac{\partial P_G}{\partial y_l} + 1_{l=G}) + (P^f + y)\frac{\partial x^1}{\partial y_l} + (P^f - K)\frac{\partial s^1}{\partial y_l}\right)$$
$$= \alpha^2 u''\left(\delta + (P^f + y)\,x^{f,2}\right)\left(x^{f,2}(\frac{\partial P_G}{\partial y_l} + 1_{l=G}) + (P^f + y)\frac{\partial x^2}{\partial y_l} + (P^f - K)\frac{\partial s^2}{\partial y_l}\right).$$

Note that the derivation of (23) and (24) wrt to y_l, $l = B, G$, implies that $\frac{\partial x^1}{\partial y_l} = -\frac{\partial x^2}{\partial y_l}$ and $\frac{\partial s^1}{\partial y_l} = -\frac{\partial s^2}{\partial y_l}$. Dividing both equations by $\alpha^1\left(u'\left(\delta + (P^f + y)\,x^{f,1}\right) - 1\right) = \alpha^2\left(u'\left(\delta + (P^f + y)\,x^{f,2}\right) - 1\right)$ and after some algebra, we deduce using the definition (35) of κ that:

$$\eta^1\,(P^f + y)\frac{\partial x^1}{\partial y_l} = -\eta^2(P^f + y)\frac{\partial x^2}{\partial y_l} = -\kappa(\frac{\partial P_B}{\partial y_l} + 1_{l=B}), \tag{37}$$

$$\eta^1(P^f + y)\frac{\partial x^1}{\partial y_l} + \eta^1(P^f - K)\frac{\partial s^1}{\partial y_l} = -\eta^2(P^f + y)\frac{\partial x^2}{\partial y_l} - \eta^2(P^f - K)\frac{\partial s^2}{\partial y_l} = -\kappa(\frac{\partial P_G}{\partial y_l} + 1_{l=G}). \tag{38}$$

D.2 Prices

Differentiating (14) with respect to y_l in the vicinity of the riskless equilibrium yields:

$$\frac{\partial P_k}{\partial y_l} = \beta \widehat{M} \sum_{j=B,G} \pi_{k,j} \left(\frac{\partial P_j}{\partial y_l} + 1_{j=l} \right),$$

$$\text{with: } \widehat{M} = 1 + \alpha^1 (u'(\delta + (P^f + y)x^{f,1}) - 1) \times (1 - \frac{V}{\eta^1 \eta^2} \frac{u''\left(\delta + (P^f + y)x^{f,1}\right)}{u'\left(\delta + (P^f + y)x^{f,1}\right) - 1} \frac{u''\left(\delta + (P^f + y)x^{f,2}\right)}{u'\left(\delta + (P^f + y)x^{f,2}\right) - 1}), \tag{39}$$

Denoting $\widetilde{M} = \frac{\beta \widehat{M}}{(1-\beta\widehat{M})(1-(\pi_{GG}+\pi_{BB}-1)\beta\widehat{M})}$, we obtain that:

$$\begin{bmatrix} \frac{\partial P_G}{\partial y_l} \\ \frac{\partial P_B}{\partial y_l} \end{bmatrix} = \widetilde{M} \begin{bmatrix} \left(\pi_{GG} - \beta\widehat{M}(\pi_{GG} + \pi_{BB} - 1)\right) 1_{l=G} + (1 - \pi_{GG})\, 1_{l=B} \\ (1 - \pi_{BB})\, 1_{l=G} + \left(\pi_{BB} - \beta\widehat{M}(\pi_{GG} + \pi_{BB} - 1)\right) 1_{l=B} \end{bmatrix} > 0. \tag{40}$$

Differentiating equation (22) with respect to y_l finally yields:

$$\frac{\partial Q_k}{\partial y_l} = \beta\widehat{M}(\frac{\partial P_G}{\partial y_l} + 1_{l=G}). \tag{41}$$

D.3 Back to Proposition 4

Using (37) and (38) with Lemma 1, we deduce the impact of a mean preserving spread of dividends on security quantities (recall that $V[y]$ $(E[y])$ is the variance (mean) of the dividend process):

$$\eta^1(P^f + y)\, \frac{\partial x^1}{\partial V[y]}\bigg|_{E[y] \text{ cst}} = \kappa \frac{2 - \pi_{GG} - \pi_{BB}}{2(y_G - y_B)(1 - \pi_{GG})} \frac{1}{1 - (\pi_{GG} + \pi_{BB} - 1)\beta\widehat{M}} > 0,$$

$$\eta^1(P^f - K)\, \frac{\partial s^1}{\partial V[y]}\bigg|_{E[y] \text{ cst}} = -\kappa \frac{2 - \pi_{GG} - \pi_{BB}}{2(y_G - y_B)} \frac{1}{1 - (\pi_{GG} + \pi_{BB} - 1)\beta\widehat{M}} \frac{2 - \pi_{BB} - \pi_{GG}}{(1 - \pi_{BB})(1 - \pi_{GG})} < 0.$$

The derivatives of the asset price in (21) relative to $V[y]$ can be expressed as $(l = B, G)$:

$$\frac{2(y_G - y_B)}{2 - \pi_{GG} - \pi_{BB}}\, \frac{\partial P_l}{\partial V[y]}\bigg|_{E[y] \text{ cst}} = (1_{l=G} - 1_{l=B}) \frac{1}{1 - \pi_{ll}} \frac{(\pi_{GG} + \pi_{BB} - 1)\beta\widehat{M}}{1 - (\pi_{GG} + \pi_{BB} - 1)\beta\widehat{M}}$$

We deduce $\frac{\partial Q_G}{\partial V[y]}\Big|_{E[y]} > \frac{\partial Q_B}{\partial V[y]}\Big|_{E[y]}$ since we have from (41): $\frac{\partial Q_k}{\partial V[y]}\Big|_{E[y]} = \pi_{k,G} \frac{2-\pi_{GG}-\pi_{BB}}{2(y_G-y_B)(1-\pi_{BB})} \widetilde{M} > 0.$

E Proof of Proposition 5

We consider the evolution of prices and quantities around the symmetric equilibrium $\alpha^1 = \alpha^2 = \alpha$, where the asset prices are P^s_G and P^s_B and those of the options are, respectively, Q^s_G and Q^s_B:

$$P^s_k = \beta \sum_{j=B,G} \pi_{k,j} \left(1 + \alpha \left(u'\left(\delta + (P^s_j + y_j)\, \frac{V}{\eta^1 + \eta^2}\right) - 1\right)\right) (P^s_j + y_j),$$

$$Q^s_k = \beta \pi_{k,G} \left(1 + \alpha \left(u'\left(\delta + (P^s_G + y_G)\, \frac{V}{\eta^1 + \eta^2}\right) - 1\right)\right) (P^s_G - K).$$

E.1 Quantities

Differentiating (19) and (20) wrt α^i ($\alpha^i = 1, 2$) yields (close to the symmetric equilibrium):

$$\eta^1(P_B + y_B)\frac{\partial x^1}{\partial \alpha^i} = (1_{i=2} - 1_{i=1})\frac{\eta^1\,\eta^2}{\eta^1+\eta^2}\frac{u'\left(\delta + (P_B^s + y_B)\,\frac{V}{\eta^1+\eta^2}\right) - 1}{\alpha\, u''\left(\delta + (P_B^s + y_B)\,\frac{V}{\eta^1+\eta^2}\right)},$$

$$\eta^1(P_G + y_G)\frac{\partial x^1}{\partial \alpha^i} + \eta^1(P_G - K)\frac{\partial s^1}{\partial \alpha^i} = (1_{i=2} - 1_{i=1})\frac{\eta^1\,\eta^2}{\eta^1+\eta^2}\frac{u'\left(\delta + (P_G^s + y_G)\,\frac{V}{\eta^1+\eta^2}\right) - 1}{\alpha\, u''\left(\delta + (P_G^s + y_G)\,\frac{V}{\eta^1+\eta^2}\right)}.$$

We deduce that $\frac{\partial x^1}{\partial \alpha^1} > 0$ and $\frac{\partial s^1}{\partial \alpha^1} < 0$ whenever condition (18) holds.

E.2 Prices

We differentiate the expressions of both asset and option prices with respect to α^i ($i = 1, 2$) in the neighborhood of the symmetric equilibrium:

$$\frac{\partial P_k}{\partial \alpha^i} = \beta\,\frac{\eta^1\,1_{i=1} + \eta^2\,1_{i=2}}{\eta^1+\eta^2}\sum_{j=B,G}\pi_{k,j}\Delta_j + \beta\sum_{j=B,G}\pi_{k,j}\,M_j\,\frac{\partial P_j}{\partial \alpha^i},$$

$$\text{with: } M_j = 1 + \alpha\left(u'\left(\delta + (P_j^s + y_j)\frac{V}{\eta^1+\eta^2}\right) - 1\right)\left(1 + \frac{(P_j^s + y_j)V}{\eta^1+\eta^2}\frac{u''(\delta + (P_j^s + y_j)\frac{V}{\eta^1+\eta^2})}{u'(\delta + \frac{(P_j^s+y_j)V}{\eta^1+\eta^2}) - 1}\right)$$

$$\Delta_j = \left(u'\left(\delta + (P_j^s + y_j)\,\frac{V}{\eta^1+\eta^2}\right) - 1\right)(P_j^s + y_j).$$

Using matrix notation, we obtain after denoting $\widetilde{M}_{GB} = 1 - \beta\pi_{GG}M_G - \beta\pi_{BB}M_B + \beta^2(\pi_{GG} + \pi_{BB} - 1)M_G M_B$:

$$\widetilde{M}_{GB}\begin{bmatrix}\frac{\partial P_G}{\partial \alpha^i}\\ \frac{\partial P_B}{\partial \alpha^i}\end{bmatrix} = \beta\frac{\eta^1 1_{i=1} + \eta^2 1_{i=2}}{\eta^1+\eta^2}\begin{bmatrix}(\pi_{GG} - \beta M_B(\pi_{GG}+\pi_{BB}-1))\Delta_G + (1-\pi_{GG})\Delta_B\\ (\pi_{BB} - \beta M_G(\pi_{GG}+\pi_{BB}-1))\Delta_B + (1-\pi_{BB})\Delta_G\end{bmatrix} > 0.$$

Analogously for the option price, we have:

$$\frac{\partial Q_k}{\partial \alpha^i} = \beta\pi_{k,G}\widehat{M}_G\frac{\partial P_G}{\partial \alpha^i} + \beta\frac{\eta^1 1_{i=1} + \eta^2 1_{i=2}}{\eta^1+\eta^2}\pi_{k,G}\left(u'(\delta + (P_G^s + y_G)\frac{V}{\eta^1+\eta^2}) - 1\right)(P_G^s - K).$$

$$\text{with: } \widehat{M}_G = 1 + \alpha\left(u'(\delta + \frac{(P_G^s + y_G)V}{\eta^1+\eta^2}) - 1\right)\left(1 + \frac{(P_G^s - K)\frac{V}{\eta^1+\eta^2}u''(\delta + (P_G^s + y_G)\frac{V}{\eta^1+\eta^2})}{u'(\delta + (P_G^s + y_G)\,\frac{V}{\eta^1+\eta^2}) - 1}\right).$$

We easily deduce that $\frac{\partial Q_G}{\partial \alpha^i} > \frac{\partial Q_B}{\partial \alpha^i} > 0$, which proves the last result in Proposition 4.

F Extension to heterogeneous suboptimal production levels

In this section, we show that allowing agents to be endowed with heterogeneous levels of suboptimal production may make our results consistent with those of **?**. We assume that type-1 and type-2 agents have access to different suboptimal production levels denoted respectively δ_1 and δ_2. The rest of the setup is unchanged. As long as δ_1 and δ_2 are not too different from each other, we can still prove the existence of an equilibrium as in Proposition 1.

In this setup, we can prove the following lemma:

Lemma 2 (Portfolio holdings with heterogeneous δ) *Assume that type-1 and type-2 agents are endowed with different suboptimal production levels denoted respectively δ^1 and δ^2, as described above. If $\delta^1 > \delta^2$, there exist probabilities $\alpha^1 > \alpha^2$ such that:*

- *type-2 agents hold a greater quantity of assets than type-1 agents, i.e. $x^2 > x^1 > 0$;*

- *type-2 agents sell options to type-1:* $s^2 < 0 < s^1$.

According to Lemma 2, despite the fact that type-1 agents face a higher probability of losing their production opportunity, they hold riskier portfolios: they hold less stocks and sell call options (and thus insurance) to type-2 agents.[13]

Proof. It is rather straightforward to show pricing equations (14) and (15) generalize to ($k = B, G$ and $i = 1, 2$):

$$P_k = \beta\pi_{k,G}\left(1 + \alpha^i\left(u'\left(\delta^i + (P_G + y_G)x^i + (P_G - K)s^i\right) - 1\right)\right)(P_G + y_G) \quad (42)$$
$$+ \beta\pi_{k,B}\left(1 + \alpha^i\left(u'\left(\delta^i + (P_B + y_B)x^i\right) - 1\right)\right)(P_B + y_B),$$
$$Q_k = \beta\pi_{k,G}\left(1 + \alpha^i\left(u'\left(\delta^i + (P_G + y_G)x^i + (P_G - K)s^i\right) - 1\right)\right)(P_G - K). \quad (43)$$

Market clearing conditions (23) and (24) still hold. Equations (42) and (43) imply that equations (19) and (20) characterizing the participation of both agents types to both markets become:

$$\alpha^1\left(u'(\delta^1 + (P_B + y_B)x^1) - 1\right) = \alpha^2\left(u'(\delta^2 + (P_B + y_B)x^2) - 1\right), \quad (44)$$
$$\alpha^1\left(u'(\delta^1 + (P_G + y_G)x^1 + (P_G - K)s^1) - 1\right) = \alpha^2\left(u'(\delta^2 + (P_G + y_G)x^2 + (P_G - K)s^2) - 1\right). \quad (45)$$

Let us first assume that $\alpha^1 = \alpha^2$. Equations (44) and (45) imply:

$$x^2 - x^1 = \frac{\delta^1 - \delta^2}{P_B + y_B} > 0,$$
$$s^1 - s^2 = \frac{(P_G + y_G) - (P_B + y_B)}{P_B + y_B}(\delta^1 - \delta^2) > 0.$$

If $\alpha^1 = \alpha^2$, type-2 agents hold more stock than type-1 and sell them calls. By continuity of (44) and (45) in α^1 and α^2, we can find two values $\alpha^1 > \alpha^2$ such that $x^2 - x^1 > 0$ and $s^1 - s^2 > 0$, which concludes the proof. ■

G Extension to a three-state economy with riskless bonds

In this section, we provide a detailed presentation of an extension of our setup to a three-state economy with riskless bonds. More formally, we consider an economy similar to the one described in Section 2 of the paper. We introduce a riskless bond that pays off one unit of consumption in every state and whose price is denoted R_t at date t. The size of the risky tree is now denoted $V_X > 0$, while we assume that the net supply of bonds is denoted $V_B > 0$. We denote b_t^i the bond holdings of agent i at date t.The option is still in zero net supply. We further assume that agents

[13]However, since $\delta^1 > \delta^2$, it would be misleading to say that type-1 agents bear a higher individual risk than type-2 agents.

cannot short-sell the bond. The program of a type-i agent can be expressed as follows:

$$\max_{(c_t^i, e_t^i, x_t^i, b_t^i, (s_t^{i,h})_h)_{t\geq 0}} E_0\left[\sum_{t=0}^{\infty}\beta^t\left(u(c_t^i)-e_t^i\right)\right] \tag{46}$$

$$\text{s.t.}c_t^i + P_t\, x_t^i + R_t b_t^i + \sum_{h=1}^{H} Q_t^h s_t^{h,i} = \xi_t^i e_t^i + (1-\xi_t^i)\delta \tag{47}$$

$$+ (P_t + y_t)\, x_{t-1}^i + b_{t-1}^i + \sum_{h=1}^{H}(P_t - K_{t-1}^h)^+ s_{t-1}^{h,i} \tag{48}$$

$$c_t^i \geq 0 \text{ and } e_t^i \geq 0 \tag{49}$$

$$P_t x_t^i + R_t b_t^i + \sum_{h=1}^{H} Q_t^h s_t^{h,i} \geq 0, \tag{50}$$

$$x_t^i \geq 0,\ b_t^i \geq 0, \tag{51}$$

$$\sum_{h=1}^{l} Q_t^h (s_t^{h,i,j})^+ \leq P_t\, x_t^{i,j} + R_t b_t^i, \tag{52}$$

$$\lim_{t\to\infty}\beta^t E_0\left[u'(c_t^i)\, x_t^i\right] = \lim_{t\to\infty}\beta^t E_0\left[u'(c_t^i)\, b_t^i\right] = \lim_{t\to\infty}\beta^t E_0\left[u'(c_t^i)\, s_t^{h,i}\right]\ (h=1,\ldots,H), \tag{53}$$

$$\{x_{-1}^i, b_{-1}^i, s_{-1}^{1,i}, \ldots, s_{-1}^{H,i}, \xi_0^i, y_0\} \text{ are given.} \tag{54}$$

To express market clearing conditions, we need to adapt the definition of the distribution Λ_t^i of type-i agents that is now a function of all security holdings, including bonds, and labor status using the probability measure $\Lambda_t^i : \mathcal{B}(\mathbb{R})^{2+H} \times \mathcal{B}(E^t) \to [0,1]$. This probability measure can be interpreted as follows: $\Lambda_t^i(X, B, S^1, \ldots, S^H, I)$ (with $(X, B, S^1, \ldots, S^H, I) \in \mathcal{B}(\mathbb{R})^{2+H} \times \mathcal{B}(E^t)$) is the measure of agents of type i, with stock holdings $x \in X$, bond holdings $b \in B$, option positions $s^h \in S^h$ $(h = 1, \ldots, H)$, and with an individual history $\xi \in I$. Using these probability measures, market-clearing conditions become:

$$\sum_{i=1,2}\int_{\mathbb{R}^{2+H}\times E^t} x\Lambda_t^i(dx, db, ds^1, \ldots, ds^H, d\xi) = V_X, \tag{55}$$

$$\sum_{i=1,2}\int_{\mathbb{R}^{2+H}\times E^t} b\Lambda_t^i(dx, db, ds^1, \ldots, ds^H, d\xi) = V_B, \tag{56}$$

$$\sum_{i=1,2}\int_{\mathbb{R}^{2+H}\times E^t} s^h\Lambda_t^i(dx, db, ds^1, \ldots, ds^H, d\xi) = 0\ (h=1,\ldots,H). \tag{57}$$

The equilibrium is defined very similarly to Definition 1:

Definition G.1 (Sequential competitive equilibrium in an economy with bonds) *A sequential competitive equilibrium is a collection of consumption and effort levels $(c_t^i, e_t^i)_{t\geq 0}$, of stock demands $(x_t^i)_{t\geq 0}$, of bond demands $(b_t^i)_{t\geq 0}$, of derivative demands $(s_t^{1,i}, \ldots, s_t^{H,i})_{t\geq 0}$ for $i = 1, 2$ and of security prices $(P_t, R_t, Q_t)_{t\geq 0}$ such that for an initial distribution of security holdings, and of idiosyncratic and aggregate shocks $\{(x_{-1}^i, b_{-1}^i, s_{-1}^{1,i} \ldots, s_{-1}^{H,i}, \xi_0^i)_{i=1,2}, y_0\}$, we have:*

1. *Individual strategies solve the optimization program (46) when prices are given;*
2. *Security prices adjust such that security markets clear at all dates and equations (55)–(57) hold;*
3. *The evolution of the probability measures Λ_t^1 and Λ_t^2 is consistent with individual choices.*

Provided that security volumes V_X and V_B are not too large, that heterogeneity remains limited and that condition (11) still holds, we can prove the existence of a limited-heterogeneity equilibrium characterized by the set of quantities and prices $(x_k^i, b_k^i, s_k^i, P_k, R_k, Q_k)_{k=B,G}^{i=1,2}$ solving the following equations ($k = 1, \dots, n$, $h = 1, \dots, H$ and $i = 1, 2$):[14]

$$
\begin{aligned}
P_k &= \beta \sum_{j=1}^{n} \pi_{k,j} (1 + \alpha^i (u'(\delta + (P_j + y_j)\, x_k^i + b_k^i + \sum_{h=1}^{H} (P_j - K_k^h)^+ s_k^{h,i}) - 1))(P_j + y_j) \\
R_k &= \beta \sum_{j=1}^{n} \pi_{k,j} (1 + \alpha^i (u'(\delta + (P_j + y_j)\, x_k^i + b_k^i + \sum_{h=1}^{H} (P_j - K_k^h)^+ s_k^{h,i}) - 1)) \\
Q_k^h &= \beta \sum_{j=1}^{n} \pi_{k,j} (1 + \alpha^i (u'(\delta + (P_j + y_j)\, x_k^i + b_k^i + \sum_{l=1}^{H} (P_j - K_k^l)^+ s_k^{l,i}) - 1))(P_j - K_k^h)^+ \\
V_X &= \eta^1\, x_k^1 + \eta^2\, x_k^2, \\
V_B &= \eta^1\, b_k^1 + \eta^2\, b_k^2, \\
0 &= \eta^1\, s_k^{h,1} + \eta^2\, s_k^{h,2}.
\end{aligned}
$$

We further simplify the model along the lines of Section 4:[15]

1. There are three states of the world $n = 3$ that we denote G, M and L (from good too bad) and that one single call is traded ($H = 1$).
2. Aggregate states are persistent, i.e. $\pi_{GG} + \pi_{MM} + \pi_{BB} > 1$.
3. The utility function u is such that $\lim_{c \to \infty} u'(c) = 0$ and:
$$X \mapsto -X \frac{u''(\delta + X)}{u'(\delta + X) - 1} \text{ is increasing for } X \in [0, u'^{-1}(1) - \delta).$$
4. The strike K of the option is such that the option exactly pays off in the good state G of the world.

There are three non redundant securities and three states of the world. Every agent type therefore holds a security portfolio, which is independent of the state of the world. The simplified equilibrium is then characterized by 15 variables $\left\{x^1, x^2, b^1, b^2, s^1, s^2, P_G, P_M, P_B, R_G, R_M, R_B, Q_G, Q_M, Q_B\right\}$ together with the following eight equations:

$$\alpha^1 \left(u'(\delta + (P_B + y_B)x^1 + b^1) - 1\right) = \alpha^2 \left(u'(\delta + (P_B + y_B)x^2 + b^2) - 1\right) \tag{58}$$

$$\alpha^1 \left(u'(\delta + (P_M + y_M)x^1 + b^1) - 1\right) = \alpha^2 \left(u'(\delta + (P_M + y_M)x^2 + b^2) - 1\right) \tag{59}$$

$$\alpha^1 \left(u'(\delta + (P_G + y_G)x^1 + b^1 + (P_G - K)s^1) - 1\right) = \alpha^2 \left(u'(\delta + (P_G + y_G)x^2 + b^2 + (P_G - K)s^2) - 1\right) \tag{60}$$

$$P_k = \beta \sum_{j=B,M,G} \pi_{k,j} \left(1 + \alpha^1 \left(u'\left(\delta + (P_j + y_j)x^1 + b^1 + 1_{j=G}(P_G - K)s^1\right) - 1\right)\right)(P_j + y_j) \tag{61}$$

$$R_k = \beta \sum_{j=B,M,G} \pi_{k,j} \left(1 + \alpha^1 \left(u'\left(\delta + (P_j + y_j)x^1 + b^1 + 1_{j=G}(P_G - K)s^1\right) - 1\right)\right) \tag{62}$$

$$Q_k = \beta \pi_{k,G} \left(1 + \alpha^1 \left(u'\left(\delta + (P_G + y_G)x^1 + b^1 + (P_G - K)s^1\right) - 1\right)\right)(P_G - K),\ k = G, B \tag{63}$$

$$V_X = \eta^1\, x^1 + \eta^2\, x^2 \tag{64}$$

$$V_B = \eta^1\, b^1 + \eta^2\, b^2 \tag{65}$$

$$0 = \eta^1\, s^1 + \eta^2\, s^2 \tag{66}$$

We can now state a proposition similar to Proposition 3 about portfolio compositions.

Proposition G.3 (Agents' portfolios in a three-state economy) *Type-1 agents, facing a greater risk of becoming unproductive, choose to hold a less risky portfolio than type-2 agents. More precisely:*

[14] We do not provide an explicit proof but it would be straightforward to follow the same lines as in Section 1 of the Appendix.

[15] The main difference with our initial setup is that we require $u'(\infty) = 0$.

- *type-1 agents hold more bonds than type-2 agents, i.e.* $b^1 > b^2 > 0$*;*
- *type-1 agents always sell call options to hedge their stock holdings.*

Moreover, if we further assume $x \mapsto -\frac{u''(x)}{u'(x)}$ *to be weakly decreasing, we obtain that in the neighborhood of* $V_X = 0$*, we have* $x^1 \geq x^2$*.*

High-risk agents use securities to hold less risky portfolios than low-risk agents. Indeed, high-risk agents hold more bonds, which provide insurance in the bad state of the world and they sell calls, which allow smoothing out payoffs of stock holdings. The position in stocks of high-risk agents can be smaller or greater than the one of low-risk agents. The intuition is the following. Bonds are mainly purchased to provide insurance in the bad state, but they also payoff in the medium state. If bonds provide insufficient insurance in medium state, high-risk agents will need to further purchase stocks and their holdings will be larger than the ones of low-risk agents. Conversely, if bonds provide "too much" insurance in medium state, type-1 agents will hold less stocks than type-2. We prove that this latter case never holds when few stocks are available and when agents have DARA utility.

Portfolio compositions in the extended economy with three states and riskless bonds are therefore very consistent with our findings of Proposition 3: high-risk agents hold less risky porfolios than low-risk agents.

Proof. We denote $u'_{j,k} = u'(\delta + (P_j + y_j)x^k + b^k)$ and $u''_{j,k} = u''(\delta + (P_j + y_j)x^k + b^k)$ for $k = 1, 2$ and $j = B, M, G$. We consider the two equations (58) and (58) together with (64) and (65) in $s^1$and x^1. Values $P_M + y_M$ and $P_B + y_B$ are considered as distinct fixed parameters. Parameters α^1 and α^2 can be varied. Remark that for $\alpha^1 = \alpha^2$, we have $x^1 = x^2$ and $b^1 = b^2$. Computing the derivative of (58) and (58) with respect to α^1 yields

$$(P_j + y_j)\frac{\partial x^1}{\partial \alpha^1} + \frac{\partial b^1}{\partial \alpha^1} = -\frac{u'_{j,1} - 1}{\alpha^1 u''_{j,1} + \alpha^2 u''_{j,2}} > 0$$

and

$$\left(\frac{1}{P_B + y_B} - \frac{1}{P_M + y_M}\right)\alpha^1 \frac{\partial b^1}{\partial \alpha^1} = \frac{1}{\frac{-(P_B+y_B)u''_{B,1}}{u'_{B,1}-1} + \frac{-(P_B+y_B)u''_{B,2}}{u'_{B,2}-1}} - \frac{1}{\frac{-(P_M+y_M)u''_{M,1}}{u'_{M,1}-1} + \frac{-(P_M+y_M)u''_{M,2}}{u'_{M,2}-1}}.$$

Let us assume $(P_M + y_M) > (P_B + y_B)$ (note that if we make the reverse assumption, the result still holds; both inequalities below will be reversed). We have using Assumption (18) about utility shape:

$$(\frac{1}{P_B + y_B} - \frac{1}{P_M + y_M})\alpha^1 \frac{\partial b^1}{\partial \alpha^1} = -\frac{u'_{M,1} - 1}{\alpha^1 u''_{M,1} + \alpha^2 u''_{M,2}} + -\frac{u'_{B,1} - 1}{\alpha^1 u''_{B,1} + \alpha^2 u''_{B,2}}$$

which implies

$$\left(\frac{1}{P_B + y_B} - \frac{1}{P_M + y_M}\right)\alpha^1 \frac{\partial b^1}{\partial \alpha^1} > 0.$$

Therefore $(P_M + y_M) > (P_B + y_B)$ implies $\frac{\partial b^1}{\partial \alpha^1} > 0$. Market clearing implies then that $b^1 > b^2$, which proves the first part of the proof.

Regarding s^1, we define the following function of s^1:

$$\psi(s^1) = \alpha^1 \left(u'(\delta + (P_G + y_G)x^1 + b^1 + (P_G - K)s^1) - 1\right)$$
$$- \alpha^2 \left(u'(\delta + (P_G + y_G)x^2 + b^2 - (P_G - K)\frac{\eta^1}{\eta^2}s^1) - 1\right),$$

which is a strictly decreasing function of s^1. To prove that $s^1 < 0$, it is sufficient to prove that $\psi(0) < 0$. To do so, let us consider $\tilde{\psi}_0 : \pi \mapsto \alpha^1 \left(u'(\delta + \pi x^1 + b^1) - 1\right) - \alpha^2 \left(u'(\delta + \pi x^2 + b^2) - 1\right)$. The function $\tilde{\psi}_0$ admits at most two zeros, which are $P_B + y_B$ and $P_M + y_M$. Therefore, since $P_G + y_G > P_B + y_B, P_M + y_M$ (since the option only pays off in the best state by construction),

$\tilde{\psi}_0(P_M + y_M)$ has the sign as $\alpha_2 - \alpha_1 < 0$. We deduce that $\tilde{\psi}_0(P_M + y_M) = \psi(0) < 0$, which implies $s^1 < 0$. This proves the second part of the proof.

If $|V_X| \ll 1$, we obtain developing (58) and (58) at the first order in x^1 and x^2:

$$-x^1 \frac{u''(\delta + b^1)}{u'(\delta + b^1)} = -x^2 \frac{u''(\delta + b^2)}{u'(\delta + b^2)}.$$

Since $b^1 > b^2$ and if we further assume $x \mapsto -\frac{u''(x)}{u'(x)}$ to be decreasing, we obtain $0 < -\frac{u''(\delta+b^1)}{u'(\delta+b^1)} \leq -\frac{u''(\delta+b^2)}{u'(\delta+b^2)}$ and $x^1 \geq x^2$, which terminates the proof. ■