Should we increase or decrease public debt? Optimal fiscal policy with heterogeneous agents*

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PRELIMINARY, COMMENTS WELCOME

Abstract

We analyze optimal fiscal policy in a heterogeneous-agent model with capital accumulation and aggregate shocks, where the government uses public debt, capital tax and progressive labor tax to finance public spending. First, the existence of a steady-state equilibrium is proven to depend on three conditions, which have insightful economic interpretations: a Laffer condition, a Blanchard-Kahn condition and a Straub-Werning condition. We show that the equilibrium can feature both a positive level of public debt and a positive capital tax at the steady state, which corrects for non-optimal private savings. In addition, after a positive public spending shock of a given net present value, the optimal public debt increases when the persistence of the shock is low, whereas it decreases when its persistence is high, due to a tradeoff between consumption smoothing and the reduction of distortions. We show that our results hold in a tractable model that can be analytically solved and in a quantitative heterogeneous-agent model, where the optimal dynamics of the whole set of fiscal tools is analyzed. The general model also provides new results on optimal tax progressivity and the dynamics of labor tax.

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1 Introduction

What is the optimal level of public debt? Should it increase or decrease when public spending is increasing? After a positive public spending shock, should the government temporarily increase capital tax or other distorting taxes, affecting the progressivity of the tax system? These old questions are likely to stay relevant in the coming years in many countries, as additional public spending for climate change or for military spending are often discussed among policy makers. Such questions require considering both distorting and redistributive effects of tax changes, while accounting for general equilibrium effects. Heterogeneous-agent models in the tradition of the Bewley-Huggett-Imohoroglu-Aiyagari literature (Bewley, 1983; Imrohoroğlu, 1989; Huggett, 1993; Aiyagari, 1994; Krusell and Smith, 1998) are relevant tools to analyze such questions, as they generate a realistic amount of heterogeneity together with general and dynamic equilibrium effects. However, after seminal papers investigating optimal fiscal policy in these environments (Aiyagari, 1995 and Aiyagari and McGrattan, 1998), the literature has moved toward a positive analysis (such as Floden, 2001; Heathcote, 2005; Rohrs and Winter, 2017, and Ferriere and Navarro, 2020, among many others) or normative policy in steady-state environments (Conesa et al., 2009) due to methodological difficulties to solve for optimal policies in heterogeneous-agent models with aggregate shocks.

This paper analyses optimal fiscal policy in heterogeneous-agent models, considering capital accumulation, progressive labor income taxation as in Heathcote et al. (2017), capital tax, public debt, and aggregate shocks. The only frictions considered are incomplete markets for idiosyncratic risk, credit constraints (which appear to be the main friction), and a limited set of fiscal instruments that could be used to provide insurance or raise resources. In particular, the planner cannot use lump-sum taxes, which are known to possibly restore Ricardian equivalence in some environments (Bhandari et al., 2017). Our analysis admittedly abstracts from other frictions, such as nominal rigidities or frictional labor markets, so as to identify new mechanisms, which will be present in other environments. Considering capital accumulation enables us to introduce the endogenous production of a store of value in possibly unlimited amounts and also to characterize the optimal dynamics of capital tax. We study in this framework the question of optimal Ramsey allocation, both at the steady state and in the dynamics following a public spending shock.

A first key feature of our model is that the optimal level of public debt is well defined. Although public debt is "liquidity" in the sense of Woodford (1990) (that is, a public store of value), its main function is a saving "absorber", as the private saving is too high when credit constraints are binding. This absorption is costly, as the planner must raise resources through distorting taxes to finance public debt interest payments. We show that the planner can choose both positive public debt and positive capital taxes. This first set of results confirms the claims that both capital tax and public debt can be positive in such environments, which has been recently challenged (Chen et al., 2021 in a special case and Chen et al. 2020). Importantly, we provide new results concerning the optimal dynamics of fiscal instruments. Considering a public spending shock (for a given overall net present value), we show that the dynamics of fiscal instruments depend crucially on the persistence of the shock. When the persistence is low, public debt increases and taxes and the progressivity of income tax decreases in the first quarters. This implements consumption smoothing, as the needs for additional public resources are temporary. When the persistence is high, there is a long-lasting need for additional public resources, which generates a decrease inpublic debt, while taxes and progressivity increase in the first quarter to front-load the adjustment. As a consequence, independently of the magnitude of the public spending shock, its persistence is a key driver of the optimal dynamics of public debt and of tax progressivity in economies where agents face credit constraints.

Solving for optimal policies in heterogeneous-agent models with aggregate shocks is difficult, and we present our analysis in two steps. First, we consider a special case of the general model, in which we derive analytical results for both the steady-state value of the planner's instruments and for their dynamics. The simplification is based on the assumption of deterministic income fluctuations as in Woodford (1990) and a utility function without wealth effect for the labor supply (or GHH utility function following Greenwood et al., 1988, or Diamond, 1998, for instance). Hence, the only friction is credit constraint, as there is no risk. In this environment, we show that a long-run steady-state equilibrium exists with positive capital tax and possibly positive public debt under three conditions that we name according to their interpretation: A Laffer condition, a Straub-Werning condition, and a Blanchard-Kahn condition. These three conditions have straightforward economic interpretations. The Laffer condition states that public spending should not be too high, otherwise this spending would not be sustainable, as it cannot be financed by any level of distorting taxes. The Straub-Werning condition elaborates on Straub and Werning (2020) and states that the public spending must be low enough to avoid the planner deviating from the steady state by continuously decreasing the capital stock (although they could levy enough resources at the steady state). The Blanchard-Kahn condition is a stability condition, which requires the planner not to deviate permanently from the steady state, due to diverging public resources. When public spending is high, the optimal fiscal system can exhibit both positive capital tax to mitigate the distortions of the labor tax, and a positive public debt that absorbs the excess savings.

In the second step, we show that the properties of the fiscal system identified in the tractable model remain valid in a quantitative heterogeneous-agent model. It is known that the tax system depends crucially on the social welfare function considered by the planner. As we are interested in the dynamics of the fiscal system, we first estimate a social welfare function consistent with the observed US tax system, following the *invert optimal approach,* used recently in Heathcote and Tsujiyama (2021). Once our model reproduces at the steady state the US tax system, we then compute the optimal capital tax, labor tax progressivity and public debt after public spending shocks with different persistences. We indeed find that the results of the simple model are valid in this more general setup. Additionally, the general model shows that the choice of the utility function (presenting a labor wealth effect or not) is key for the dynamics of the labor tax. When the utility function displays a labor wealth effect, the labor tax falls at impact when the persistence is low. When the utility function has no wealth effect (as with a GHH utility function), the labor tax always increases at impact independently of the persistence of the public spending shock.

Computing Ramsey optimal policies in the presence of aggregate shocks is one frontier in heterogeneous-agent models. We here follow the methodology of LeGrand and Ragot (2022), which uses a Lagrangian approach to derive the policy of the planner. This approach allows for occasionally-binding credit constraints, which appears to be the key friction for our research question. The simulation of the model is based on the truncation approach, which is shown to be accurate in Le Grand and Ragot (2022). It is also used in LeGrand et al. (2021) for the analysis of joint monetary and fiscal policy.

The paper is related to a recent and relatively thin quantitative literature studying optimal Ramsey policies in heterogeneous-agent models considering transitions (e.g., Conesa et al., 2009; Açikgöz et al., 2018; Dyrda and Pedroni, 2018; Nuño and Thomas, 2020; Bhandari et al., 2020). In this literature, LeGrand and Ragot (2022) use a Lagrangian approach (taken from Marcet and Marimon, 2019) and a truncation procedure to simulate the model. The gain is to allow for economic interpretation of the first-order conditions of the planner. The paper is also related to the literature on optimal capital taxation (Chari et al., 1994; Farhi, 2010; Chari et al., 2016; or Straub and Werning, 2020, among others), optimal tax redistribution and progressivity (e.g., Bassetto, 2014, or Heathcote et al., 2017). We consider here the implications of heterogeneity and occasionally-binding credit constraints. Finally, the paper is also connected to the literature relying on tractable models to derive optimal policies and identify new economic channels (Bilbiie, 2008; Gottardi et al., 2014; Heathcote et al., 2017; Bilbiie and Ragot, 2020; Acharya et al., 2020; Heathcote and Tsujiyama, 2021, among many others). In this literature, we find that the framework of Woodford (1990) is particularly useful to study optimal fiscal policy.

The rest of the paper is organized as follows. In Section 2, we present the general environment. We present simplifying assumptions and solve the tractable model in Section 3. The general model is then analyzed in Section 4 and simulated in Section 5. Section 6 concludes.

2 The environment

Time is discrete and indexed by $t = 0, 1, \ldots$ The economy is populated by a continuum of agents distributed along a set I with measure ℓ. We follow Green (1994) and assume that the law of large numbers holds. The economy features production and a benevolent government that raises distorting taxes to finance an exogenous stream of public spending.

2.1 Risks

The economy is plagued by an idiosyncratic risk. The aggregate shock solely affects public spending denoted by $(G_t)_{t\geq 0}$ and is therefore assimilated to a public spending shock. Furthermore, we assume that the whole path of public spending $(G_t)_{t\geq 0}$ becomes known to all agents in period 0. We will solve for the optimal adjustment of the economy after such a shock, also known as an MIT shock.[1]

Agents face an uninsurable productivity risk, denoted by y. Individual productivity levels follow independent first-order Markov chains, whose state-space is the finite set $\{y_1, \ldots, y_K\}$ and the transition matrix is denoted by Π. We assume that the Markov chain admits a stationary distribution that is denoted by the K-dimensional vector n^y, verifying $n^y = \Pi n^y$.[2] When an agent is endowed with productivity y, she will earn a before-tax labor wage $\tilde{w} y l$, where l denotes her labor supply and $\tilde{w}$ is the before-tax hourly wage. In period t, the productivity of agent i is y_t^i, whereas the whole history of shocks up to t is denoted by $y^{i,t} := \{y_0^i, ..., y_t^i\}$.

Finally, we assume that agents enter the economy at date 0 with an initial joint distribution of wealth and productivity $(a_{-1}^i, y_0^i)_i$ drawn from a distribution Λ_0.

2.2 Production

The production sector is standard. The unique consumption good of the economy is produced by a profit-maximizing representative firm. At any date t, the firm production function combines labor L_t and capital K_{t-1} – which needs to be installed one period in advance – to produce Y_t units of the consumption good. The production function is assumed to be of the Cobb-Douglas type featuring constant returns to scale and capital depreciation. The TFP is normalized to one. Formally, the production is defined as:

$$Y_t = F(K_{t-1}, L_t) = K_{t-1}^{\alpha} L_t^{1-\alpha} - \delta K_{t-1},$$

where $\alpha \in (0,1)$ is the capital share and $\delta \in (0,1)$ the capital depreciation rate.

The firm rents labor and capital at respective factor prices $\tilde{w}_t$ and $\tilde{r}_t$. The profit maximization conditions of the firm imply the following expressions for factor prices:

$$\tilde{w}_t = F_L(K_{t-1}, L_t) \text{ and } \tilde{r}_t = F_K(K_{t-1}, L_t). \quad (1)$$

2.3 Assets

In addition to capital, the economy also features public debt, whose size is denoted by B_t in period t. Public debt consists of one-period bonds issued by a benevolent government, which are

[1] It is known that one can derive a first-order approximation of the dynamics of the model in the presence of aggregate shocks, using the information obtained from MIT shocks (Boppart et al., 2018, Auclert et al., 2019).

[2] In the quantitative analysis of Section 5, the Markov chain can be shown to be irreducible and aperiodic – hence n^y exists and is unique.

assumed default-free. In the absence of aggregate risk in this economy, both capital and public debt are perfect substitutes, and no-arbitrage implies that they must pay the same after-tax return. Agents thus trade shares of a riskless asset, whose aggregate supply is the sum of public debt and capital. They face a credit limit and cannot borrow more than an exogenous amount denoted $\overline{a} \geq 0$.

2.4 Government

A benevolent government has to finance the exogenous stream of public spending $(G_t)_{t\geq 0}$ by levying distorting taxes on capital and labor and issuing public debt. The tax on capital is linear, with a rate $(\tau_t^K)_{t\geq 0}$. The tax on labor income is assumed to be non-linear, and possibly time-varying. We denote by $T_t(\tilde{w}yl)$ the amount of labor tax paid at date t by an agent earning the labor income $\tilde{w}yl$ by supplying l hours at a wage rate $\tilde{w}$ and a productivity y. We follow Heathcote et al. (2017) (henceforth, HSV) and consider the following functional form:

$$T_t(\tilde{w}yl) := \tilde{w}yl - \kappa_t(\tilde{w}yl)^{1-\tau_t}, \tag{2}$$

where κ_t captures the level of labor taxation and τ_t the progressivity. Both parameters are assumed to be time-varying and will be the planner's instruments. When $\tau_t = 0$, labor tax is linear with rate $1 - \kappa_t$. Oppositely, the case $\tau_t = 1$ corresponds to full income redistribution, where all agents earn the same post-tax income κ_t. Functional form (2), combined with the linear capital tax, allows one to realistically reproduce the actual US system and its progressivity (see Heathcote et al., 2017 or Ferriere and Navarro, 2020).[3]

Using the public debt description of Section 2.3, the government budget constraint can thus be written as:

$$G_t + (1 + \tilde{r}_t)B_{t-1} = \int T_t(\tilde{w}_t y^i l_t^i)\ell(di) + \tau_t^K \tilde{r}_t(B_{t-1} + K_{t-1}) + B_t. \tag{3}$$

To simplify the government budget constraint, we introduce in the spirit of Chamley (1986), generalized post-tax factor prices, which are denoted without a tilde. We define the gross and net interest rates r_t and R_t, as well as the wage rate w_t, as follows:

$$w_t := \kappa_t(\tilde{w}_t)^{1-\tau_t}, \tag{4}$$

$$R_t := 1 + r_t = 1 + (1 - \tau_t^K)\tilde{r}_t. \tag{5}$$

The model can analytically be expressed using the pair of post-tax rates (R_t, w_t) rather than pre-tax ones $(\tilde{r}_t, \tilde{w}_t)$. This considerably simplifies the model exposition and its tractability. The

[3] The literature uses either the combination of a linear tax and of a lump-sum transfer (e.g., Dyrda and Pedroni, 2018, Açikgöz et al., 2018) or the HSV structure. Heathcote and Tsujiyama (2021) show that the HSV structure is quantitatively more relevant. Opting for the HSV tax structure enables us to discuss the dynamics of optimal tax progressivity, following a public spending shock.

values of the fiscal instruments τ_t^K, κ_t, and τ_t can then be recovered from the allocation.

With the post-tax notation and taking advantage of the property of homogeneity of the production function, we deduce that the governmental budget constraint (3) can also be written as follows:

$$G_t + R_t B_{t-1} + (R_t - 1)K_{t-1} + w_t \int_i (y_t^i l_t^i)^{1-\tau_t} \ell(di) = F(K_{t-1}, L_t) + B_t. \tag{6}$$

2.5 Agents' program and resource constraints

At each date t, agents consume a unique good in quantity c_t, supply labor in quantity l_t, and save an amount a. They derive an instantaneous utility from consumption and labor supply denoted by $U(c_t, l_t)$. The utility function will be specified later on. Agents are expected utility maximizers with standard additive intertemporal preferences. The discount factor is constant and denoted $\beta \in (0, 1)$. Agents maximize at date 0 the expected discounted value of future utilities, equal to $\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t U(c_t, l_t)\right]$, where $\mathbb{E}_0$ is the unconditional expectation over the aggregate risk and over the agent's own idiosyncratic risk.

When choosing their plans for consumption $(c_t)_{t\geq 0}$, labor supply $(l_t)_{t\geq 0}$, and savings $(a_t)_{t\geq 0}$ to maximize their expected utility, agents face two constraints: (i) a budget constraint, and (ii) a credit constraint. Their budget constraints state that agents' consumption and savings should be financed solely out of net labor income and net capital income. Using the post-tax rate definition (4), the post-tax labor income amounts to $\tilde{w}_t y_t^i l_t^i - T_t(\tilde{w}_t y_t^i l_t^i) = w_t (y_t^i l_t^i)^{1-\tau_t}$ for an agent supplying labor l_t^i with productivity y_t^i. The post-tax capital income is equal to $R_t a_{t-1}^i$ for an agent with beginning-of-period wealth a_{t-1}^i. Formally, the program of an agent i endowed with the given initial wealth a_{-1}^i can be expressed as:

$$\max_{\{c_t^i, l_t^i, a_t^i\}_{t\geq 0}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t^i, l_t^i), \tag{7}$$

$$c_t^i + a_t^i = R_t a_{t-1}^i + w_t (y_t^i l_t^i)^{1-\tau_t}, \tag{8}$$

$$a_t^i \geq -\underline{a},\ c_t^i > 0,\ l_t^i > 0. \tag{9}$$

The solution of the previous program is a set of policy rules $c_t : \mathcal{Y}^t \times \mathbb{R} \to \mathbb{R}^+$, $a_t : \mathcal{Y}^t \times \mathbb{R} \to [-\bar{a}; +\infty)$ and $l_t : \mathcal{Y}^t \times \mathbb{R} \to \mathbb{R}^+$ which determine consumption, saving and labor supply decisions as a function of the idiosyncratic history y_i^t of agent i and of her initial wealth a_{-1}^i. However, to lighten the notation, we will simply write c_t^i, a_t^i and l_t^i (instead of $c_t(y_i^t, a_{-1}^i)$, $a_t(y_i^t, a_{-1}^i)$ and $l_t(y_i^t, a_{-1}^i)$). We will use the same notation for all variables, as summarized by the next remark.

Remark 1 (Simplifying Notation) *If an agent has an idiosyncratic history y_i^t, and initial wealth a_{-1}^i at period t, we will then denote by X_t^i the realization in state (y_i^t, a_{-1}^i) of any random variable $X_t : \mathcal{Y}^t \times \mathbb{R} \to \mathbb{R}$*

A consequence of Remark 1 is that the aggregation of the variable X_t in period t over the whole agent population will be written as $\int_i X_t^i \ell(di)$, instead of the more involved explicit notation $\int_{a_{-1}} \sum_{y^t \in \mathcal{Y}^t} \theta_t(y^t) X(y^t, a_{-1}) d\Lambda_0(a_{-1}, y_0)$.

Denoting by $\beta^t \nu_t^i \geq 0$ the Lagrange multiplier on the agent's credit constraint, the consumption Euler equation can be written as:

$$U_c(c_t^i, l_t^i) = \beta \mathbb{E}_t \left[R_t U_c(c_{t+1}^i, l_{t+1}^i) \right] + \nu_t^i, \tag{10}$$

where U_c and U_l denote the derivatives of U with respect to the first and second variables, respectively. Note that, because of our assumption of MIT shocks, the expectation operator in (10) as well as in the rest solely concerns idiosyncratic shocks.

The labor Euler equation yields:

$$-U_l(c_t^i, l_t^i) = (1 - \tau_t) w_t (y_t^i l_t^i)^{-\tau_t} U_c(c_t^i, l_t^i). \tag{11}$$

The clearing of financial and labor markets implies the following equalities:

$$A_t = K_t + B_t \text{ and } \int y_t^i l_t^i \ell(di) = L_t. \tag{12}$$

The clearing of the goods market reflects that the sum of aggregate consumption, public spending and new capital stock balances the output production and past capital:

$$\int_i c_t^i \ell(di) + G_t + K_t = K_{t-1} + F(K_{t-1}, L_t). \tag{13}$$

We can now formulate our definition of a sequential equilibrium in this economy.

Definition 1 (Competitive equilibrium) *A competitive equilibrium is a collection of individual variables $(c_t^i, l_t^i, a_t^i)_{t \geq 0, i \in \mathcal{I}}$, of aggregate quantities $(K_t, L_t, Y_t)_{t \geq 0}$, of price processes $(\tilde{w}_t, \tilde{r}_t)_{t \geq 0}$, of fiscal policy $(\tau_t^K, \kappa_t, \tau_t, B_t)_{t \geq 0}$ and of public spending $(G_t)_{t \geq 0}$ such that, for an initial distribution of wealth and productivity $(a_{-1}^i, y_0^i)_{i \in \mathcal{I}}$, and for initial values of capital stock and public debt verifying $K_{-1} + B_{-1} = \int_i a_{-1}^i \ell(di)$, we have:*

1. *given prices, individual strategies $(c_t^i, l_t^i, a_t^i)_{t \geq 0, i \in \mathcal{I}}$ solve the agent's optimization program in equations (7)–(9);*
2. *financial, labor, and goods markets clear: for any $t \geq 0$, equations (12) and (13) hold;*
3. *the government budget is balanced: equation (3) holds for all $t \geq 0$;*
4. *the pre-tax factor prices $(\tilde{w}_t, \tilde{r}_t)_{t \geq 0}$ are consistent with the firm's program (1).*

A stationary equilibrium is a competitive equilibrium where all aggregate variables have converged toward constant values.

2.6 The Ramsey equilibrium

The first step to consider optimal policy design is to choose a Social Welfare Function (SWF, henceforth). We assume that the planner considers a weighted sum of agents' utilities, where the agent's weight at date t depends on their current productivity: $\omega(y_t^i)$.[4] This specification is similar to the approach in Heathcote and Tsujiyama (2021) and nests the standard utilitarian SWF, that we use in Section 3 and corresponds to equal weights. This more general SWF also allows us to reproduce the US fiscal system in the quantitative Section 5. Formally, the SWF that corresponds to the planner's aggregate welfare criterion can be expressed as:

$$W_0 = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \int_i \omega(y_t^i) U(c_t^i, l_t^i) \ell(di) \right]. \tag{14}$$

The Ramsey program consists in finding the fiscal policy that corresponds to the competitive equilibrium with the highest aggregate welfare for the SWF under consideration. This problem is difficult. The labor tax directly affects the labor supply, the capital tax directly affects the saving incentives, public debt directly affects the capital stock for a given total private saving. All these instruments have indirect general equilibrium effects on prices and thus on the welfare of heterogeneous agents. As mentioned in the introduction, the existence of stationary equilibria with strictly positive values for the instrument is an open question. We first provide a characterization in a simple environment, before presenting the analysis in the general case.

3 Analytical results in the simple model

3.1 Model specification

To obtain a tractable framework we make three additional assumptions. These assumptions are only valid in this analytical analysis and will be relaxed in the quantitative analysis of Section 5. The first assumption concerns the functional form of labor taxes, which are assumed to be linear.

Assumption A *We assume that the labor tax is linear. Formally, we set in (2)* $\tau_t = 0$ *and denote* $\tau_t^L := 1 - \kappa_t$*, such that:*

$$T_t(\tilde{w}yl) := \tau_t^L \tilde{w}yl.$$

Our second assumption is about the utility function.

[4] A more general specification would consider the weights being a function of the whole history of idiosyncratic shocks for each agent: $\omega_t(y^{i,t})$. As such a generalization is not necessary in the quantitative analysis, we hence follow the simpler formulation.

Assumption B *We assume that the instantaneous utility function U is of the GHH-type:*

$$U(c,l) := \log\left(c - \chi^{-1}\frac{l^{1+1/\varphi}}{1+1/\varphi}\right),$$

where $\varphi > 0$ is the Frisch elasticity of labor supply, and $\chi > 0$ scales labor disutility.

Assumption B, which is used in Diamond (1998) among others, simplifies the algebra for the Ramsey program by avoiding wealth effects for the labor supply. The log function also simplifies the computations.

The third assumption is about the productivity process.

Assumption C *We assume that there are only two productivity levels, equal to zero and one respectively: $\mathcal{Y} = \{0, 1\}$. Furthermore, the transition matrix is anti-diagonal:*

$$\Pi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{15}$$

while the initial distribution is such that: (i) a mass one of agents have productivity 1 with an identical beginning-of-period wealth; and (ii) a mass one of agents have productivity 0 with an identical beginning-of-period wealth (but possibly different from the one of employed agents).

The main implication of Assumption C is to simplify the equilibrium wealth distribution. First, there are only two productivity levels. The first corresponds to a null productivity, and hence to a null labor supply. This zero productivity state will be called unemployment. The other productivity level is normalized to one and will correspond to employment. Second, equation (15) implies that the transitions to and out of unemployment are deterministic. Currently unemployed agents become employed in the next period and the other way around. Coupled with the assumption regarding the initial wealth distribution, Assumption C implies that at any date, the equilibrium features only two types of agents and two wealth levels. Our setup is thus similar to the one of Woodford (1990), in which two agents switch deterministically between employment and unemployment. For the sake of simplicity, the two types of agents will be called according to their current employment status: "employed" (subscript e) and "unemployed" (subscript u).

The fourth and last (not restrictive) assumption is about the credit constraint.

Assumption D *The credit-constraint is set to zero: $\underline{a} = 0$.*

Structure of the equilibrium. Taking advantage of Assumptions B–D, the model is highly tractable and equilibrium characterization is quite easy to derive. Using the peculiar equilibrium structure, the individual budget constraints (8) become:

$$c_{e,t} + a_{e,t} = R_t a_{u,t-1} + w_t y_t l_{e,t}, \tag{16}$$

$$c_{u,t} + a_{u,t} = R_t a_{e,t-1}, \tag{17}$$

for employed (subscript e) and unemployed (subscript u), respectively. Note that the definitions (4) and (5) of the post-tax rates R_t and w_t are still valid (with $\tau_t = 0$ and $\kappa_t = 1 - \tau_t^L$). We can already state a first result regarding employed agents.

Result 1. *In any equilibrium, employed agents cannot be credit-constrained at any date.*

This is a direct consequence of budget constraint (17) with $c_{u,t} > 0$. Should we have $a_{e,t} = 0$ at some date t, we would have $c_{u,t+1} = -a_{u,t+1} \leq 0$, which would contradict the consumption strict positivity constraint. A consequence of Result 1. is that we have only two possible types of (steady-state) equilibria: one in which unemployed agents are not constrained, and one in which they are.

Taking advantage of the GHH property of the utility function and of the linearity of labor taxes, the labor Euler equation (11) for employed agents simplifies into:

$$l_{e,t} = (\chi w_t)^{\varphi}, \tag{18}$$

which depends only on the hourly wage w_t. The labor and financial market clearing conditions become in this set-up:

$$L_t = l_{e,t} \text{ and } B_t + K_t = a_{e,t} + a_{u,t}. \tag{19}$$

The governmental budget constraint (6) can be simplified using (18) and (19) as follows:

$$\begin{aligned} G_t + B_{t-1} + (R_t - 1)(a_{e,t-1} + a_{u,t-1}) + w_t(\chi w_t)^{\varphi} = \\ B_t + F(A_t, a_{e,t-1} + a_{u,t-1} - B_{t-1}, (\chi w_t)^{\varphi}). \end{aligned} \tag{20}$$

Finally, using budget constraints (16) and (17) and labor Euler equation (18), we deduce that Euler equations for consumption (10) can be expressed as:

$$\beta \left(R_t a_{u,t-1} - a_{e,t} + \frac{w_t(\chi w_t)^{\varphi}}{1+\varphi} \right) = a_{e,t} - a_{u,t+1}/R_{t+1}, \tag{21}$$

$$R_{t+1} a_{u,t} - a_{e,t+1} + \frac{w_{t+1}(\chi w_{t+1})^{\varphi}}{1+\varphi} \geq \beta R_{t+1}(R_t a_{e,t-1} - a_{u,t}), \tag{22}$$

where the first Euler equation holds with equality at all dates as a consequence of Result 1.. Expectations have been dropped from Euler equations due to MIT shocks and the deterministic transition between income levels.

We assume that the planner uses a utilitarian SWF and maximizes the aggregate welfare of the economy where all agents have the same weight. The planner's aggregate welfare criterion can be expressed as:

$$W_0 = \sum_{t=0}^{\infty} \beta^t \left[\log(c_t^u) + \log\left(c_t^e - \chi^{-1} \frac{l_{e,t}^{1+1/\varphi}}{1+1/\varphi} \right) \right], \tag{23}$$

where we have used the fact that there are only two consumption levels in equilibrium. The Ramsey problem consists in selecting the fiscal instruments $(\tau_t^K, \tau_t^L, B_t)_{t\geq 0}$ that correspond to the competitive equilibrium with the highest aggregate welfare.

We now investigate the two possible types of Ramsey equilibria: one in which unemployed agents are not credit-constrained, which will correspond to the first-best allocation, and the other in which unemployed agents are (optimally) constrained.

3.2 The first-best allocation

The first-best allocation corresponds to the allocation maximizing the aggregate welfare subject to the economy-wide resource constraint – without consideration of further constraints. Formally, the planner chooses the consumption and labor paths $(c_t^e, c_t^u, l_{e,t})_t$ that maximize the objective of equation (23), subject to the resource constraint:

$$c_t^e + c_t^u + G_t + K_t = K_{t-1} + K_{t-1}^{\alpha} l_{e,t}^{1-\alpha} - \delta K_{t-1}. \tag{24}$$

The steady-state first-best allocation is then straightforward to deduce and, for space considerations, is provided in Appendix A.1. Allocation efficiency implies that the two agents will have the same marginal utility of consumption, while the marginal disutility of labor is set equal to its marginal productivity, and the marginal productivity of capital is equal to the inverse of the discount factor, $1/\beta$. We specify the first-best production level – which will be useful later on:

$$Y_{FB} \quad := \quad \chi(1-\alpha)^{\varphi} \left(\frac{\alpha}{\frac{1}{\beta} + \delta - 1} \right)^{\frac{\alpha(1+\varphi)}{1-\alpha}}. \tag{25}$$

3.3 The case of low public spending: Decentralizing the first-best equilibrium

We show here that if the level of public spending is low enough, then the planner is able to replicate the first-best allocation. The intuition is rather straightforward. When G is low enough, the planner can hold a part of the capital stock and finance the public spending out of the capital interest payments. In this case, the planner can choose null taxes and set $\tau^K = \tau^L = 0$, while implementing perfect consumption smoothing. We define the quantity

$$\overline{g}_1 := \frac{1-\beta}{\beta} \frac{\alpha}{1/\beta + \delta - 1} - \frac{1-\beta}{1+\beta} \frac{1-\alpha}{\varphi + 1}, \tag{26}$$

and state the following result.

Proposition 1 *If the public spending verifies $G \leq \overline{g}_1 Y_{FB}$, the steady-state Ramsey allocation is the first-best steady-state allocation characterized by zero taxes: $\tau^L = \tau^K = 0$, and perfect consumption smoothing.*

The proof can be found in Appendix A.2. The condition on the threshold $\overline{g}_1$ can be understood as a condition on the total capital stock. Indeed, in a production economy, the planner can levy financial resources by holding assets, i.e., a negative public debt. However, since agents cannot borrow (and hence cannot supply liquidity to the government), government asset holdings (equal to $-B$), must remain smaller than the capital stock: $-B \leq K$. This means the payoffs of the governmental asset holdings are bounded from above, and hence public spending cannot be too large and cannot exceed the capital payoffs. This constraint translates into a condition on model parameters through $\overline{g}_1$, which is increasing in the capital share α, the Frisch elasticity φ and decreasing in capital depreciation δ. The effect of a higher β is ambiguous, as on one side it increases the capital stock (because agents are more patient), but on the other side it diminishes the capital return (equal to $1/\beta$).

3.4 The equilibrium with binding credit constraint and positive capital taxation

We now turn to the only other equilibrium that admits an interior steady state.[5] To rule out the possibility of a first-best equilibrium, we make the following assumption.

Assumption E *We assume:*

$$G > \overline{g}_1 Y_{FB}.$$

We proceed by construction. This equilibrium features binding credit constraint for unemployed agents. In that case, unemployed agents hold no asset at any date: $a_{u,t} = 0$. The Euler equation (21) of employed agents implies:

$$a_{e,t} = \frac{\beta}{1+\beta} \frac{w_t(\chi w_t)^{\varphi}}{1+\varphi} > 0, \tag{27}$$

which is positive whenever $w_t > 0$. Substituting the expression (27) of $a_{e,t}$ and using $a_{u,t} = 0$, the financial market clearing condition becomes:

$$B_t + K_t = \frac{\beta}{1+\beta} \frac{w_t(\chi w_t)^{\varphi}}{1+\varphi}. \tag{28}$$

These expressions allow one to compute the optimal allocation. To simplify the Ramsey program of Section 2.6, we proceed in two steps. First, we use individual budget constraints (16) and (17) and the Euler labor equation (18) to express the Ramsey program in terms of savings choices and of the three instruments of fiscal policy $(w_t, R_t, B_t)_{t\geq 0}$ with post-tax prices, following Chamley (1986). Second, we use the savings expression (27) of employed agents (which reflects employed agents' Euler equation) and $a_{u,t} = 0$ to express the Ramsey problem solely as a function of the fiscal policy $(B_t, w_t, R_t)_{t\geq 0}$. The Ramsey planner's program can thus be written as the

[5] We explain in Section A.7 below that no other equilibrium with interior steady state exists.

maximization of a (transformed) welfare criterion subject to the governmental budget constraint:

$$\max_{(B_t,w_t,R_t)_{t\geq 0}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left((1+\beta)\log\left(\frac{1}{1+\beta}\frac{w_t(\chi w_t)^{\varphi}}{\varphi+1}\right) + \log(\beta R_t)\right), \tag{29}$$

$$\text{s.t. } G + B_{t-1} + (R_t - 1)\frac{\beta}{1+\beta}\frac{w_{t-1}(\chi w_{t-1})^{\varphi}}{1+\varphi} + w_t(\chi w_t)^{\varphi} = \tag{30}$$
$$F\left(\frac{\beta}{1+\beta}\frac{w_{t-1}(\chi w_{t-1})^{\varphi}}{1+\varphi} - B_{t-1}, (\chi w_t)^{\varphi}\right) + B_t,$$

with furthermore the Euler inequality (22) stating that the unemployed are actually credit-constrained. At the steady-state, this condition is equivalent to $\beta R < 1$ – which will always hold in this equilibrium. Two other constraints are implicit in the above program: (i) $w_t > 0$, and (ii) $R_t > 0$, which correspond to the positivity of consumption levels for employed and unemployed agents.

Before deriving first-order conditions, three important remarks are in order. First, this simple model allows for the direct expression of the planner's objective, but in more general models, this is not possible and the set of Euler equations must be used as a constraint. In this last case, one can use either the Lagrangian method used in LeGrand and Ragot (2022), or the primal approach used in Bhandari et al. (2020), if credit constraints do not occasionally bind. We show in Appendix B.1 that the first-order equations derived here are identical to the ones derived with the Lagrangian method.

Second, even in this simple framework, we need to check that the Karush–Kuhn–Tucker conditions apply to our problem, and that the first-order conditions actually characterize an equilibrium. Because of the non-linearity of the constraint (30), the standard Slater (1950) condition does not apply in our problem. We therefore need to check another constraint qualification. This is done in Appendix A.3, where we verify that the linear independence constraint qualification holds in our problem.

Third, we verify that the first-order conditions indeed characterize a maximum. This is done in Appendix A.4, where we prove that the Ramsey problem is concave.

Once these tedious though noteworthy verifications have been performed, we can be confident that the first-order conditions of the Ramsey program (29)–(30) actually characterize the optimal solution. We prove in Appendix A.5 that these FOCs can be written for all dates $t \geq 0$ as follows:

$$1 = \mu_t\left(\frac{1}{1+\beta} - \varphi\frac{\tau_t^L}{1-\tau_t^L}\right)w_t\frac{(\chi w_t)^{\varphi}}{1+\varphi}, \tag{31}$$

$$\mu_t = \beta(1+\tilde{r}_{t+1})\mu_{t+1}, \tag{32}$$

$$1 = R_t\mu_t\frac{\beta}{1+\beta}\frac{w_{t-1}(\chi w_{t-1})^{\varphi}}{1+\varphi}, \tag{33}$$

where we still denote by $\beta^t\mu_t$ the Lagrange multiplier on the governmental budget constraint and also define w_{-1} as the solution of $a_{-1} = \frac{\beta}{1+\beta}\frac{w_{-1}(\chi w_{-1})^{\varphi}}{1+\varphi}$.

Equation (31) characterizes the labor tax, while (33) characterizes the capital tax. Equation (32) is an Euler-like equation for the Lagrange multiplier on the governmental budget constraint – and does not feature any expectation operator because of MIT shocks. Although obvious, it is useful to state that prices (including the Lagrange multiplier μ on the governmental budget constraint) must be positive for the equilibrium to exist.

Result 2. *The equilibrium with binding credit constraints features the following restrictions:*

$$w_t, R_t, \mu_t > 0. \tag{34}$$

We use this result in the proof of the equilibrium existence below.

Steady-state analysis: Properties. We first characterize the steady state and then prove its existence. We will denote steady-state quantities with no subscript. For instance, R will be the steady-state gross post-tax interest rate. First, note that the restrictions of Result 3. still hold at the steady state. In particular, $\mu > 0$ implies from FOC (32) that:

$$\tilde{r} = \frac{1-\beta}{\beta}, \tag{35}$$

as in the first-best equilibrium. The previous equation is called the Modified Golden Rule from Aiyagari (1995, Proposition 1). This results comes from the fact the planner uses public debt to smooth the marginal cost of tax distortions with the same discount factor as the agents, facing no financial frictions. Its Euler equation implies this efficient steady-state value. We also have the same capital-to-labor supply ratio as in the first-best: $K/L = K_{FB}/L_{FB}$, as well as the same pre-tax wage: $\tilde{w} = w_{FB}$. $w = (1-\tau^L)w_{FB}$, and $Y = (1-\tau^L)^{\varphi}Y_{FB}$. Using $R = 1 + (1-\tau^K)\tilde{r}$ and (35), we obtain the following expression for the capital tax:

$$\tau^K = \varphi\frac{1+\beta}{1-\beta}\frac{\tau^L}{1-\tau^L}, \tag{36}$$

which is an increasing function of the labor tax.

Equation (36) is a key-equation of the simple model, which states that in equilibrium (when it exists, see below), the labor and capital taxes have a simple relationship. When levying resources, the planner faces a trade-off. One the one hand, they can raise the labor tax at the cost of reducing labor supply and GDP, while on the other hand they can raise capital tax, but affect saving incentives and capital accumulation. When the planner affects private savings, the public debt needs to adjust to balance financial markets for the Modified Golden Rule (35) to hold and the capital-to-labor ratio to verify $K/L = K_{FB}/L_{FB}$. As a consequence, the labor tax mostly affects the *production efficiency* (the level of GDP), whereas the capital tax affects *allocation efficiency* (the consumption smoothing between periods). Relationship (36) shows that in equilibrium the capital tax increases with the labor tax: both distortions increase together with the financial requirements that the planner has to finance.[6] In particular, the capital tax is positive whenever the labor tax is.

The Straub-Werning threshold. The relationship (36) does not provide any upper bound on the capital tax, which diverges when τ^L becomes close to 100%. However, the post-tax interest rate sets an implicit bound on the capital tax. Indeed, the post-tax interest rate must remain positive – otherwise the financial market would not exist any more. The positivity of the post-tax rate is equivalent to the positivity of the Lagrange multiplier μ through FOC (33). Combining the positivity of the post-tax interest rate with definition (35) of the pre-tax interest rate implies that the capital tax must remain below a threshold $\overline{\tau}^K_{SW} := \frac{1}{1-\beta}$ – where SW stands for Straub-Werning (see the discussion below). The relationship (36) equivalently implies that the labor tax must remain below a threshold $\overline{\tau}^L_{SW} := \frac{1}{1+(1+\beta)\varphi}$. FOC (31) and the positivity of μ implies the exact same bound $\overline{\tau}^L_{SW}$ on the labor tax. These two tax thresholds imply an upper bound on the level of public spending: $G < \overline{g}_{SW} Y_{FB}$, where:

$$\overline{g}_{SW} := \overline{g}_1 + (1-\alpha)\left(1 + \frac{1-\beta}{1+\beta}\frac{1}{1+\varphi} + \frac{\varphi}{1+\varphi}\right)(1-\overline{\tau}^K_{SW})^{\varphi}. \tag{37}$$

We summarize this in the following result, before the interpretation.

Result 3. *The equilibrium with a binding credit constraint exists only if $\tau^L < \overline{\tau}^L_{SW}$, $\tau^K < \overline{\tau}^K_{SW}$, or*

$$G < \overline{g}_{SW} Y_{FB}. \tag{38}$$

When the public spending is higher than the SW bound, the financing of public spending implies negative values of the steady Lagrange multiplier on the government budget constraint. Such situations cannot be ruled out and are possible for some parametrizations. However, even though a steady-state equilibrium does not exist in these situations, we show that there exists a non-stationary equilibrium, where the Lagrange multiplier on the governmental budget constraint diverges to infinity: $\mu_t \to_t \infty$ and the gross interest rate converges to zero: $R_t \to_t 0$ (see Appendix A.8). In other words, this situation is similar to the one in Straub and Werning (2020), where a stationary equilibrium does not exist but a non-stationary one does. It is noteworthy that a similar pattern emerges despite the differences between our set-ups. We have time-varying agents' types (although the switch deterministic), an endogenous credit constraint, a distorting tax on endogenous labor supply, and public debt. A key difference in our simple heterogeneous-agent model with Straub and Werning (2020) or with Lansing (1999) is that a steady-state equilibrium (including a finite Lagrange multiplier) exists for some public spending levels, even for log utilities

[6] It can be checked that τ^K/τ^L increases with the discount factor β and the Frisch elasticity.

and equal weights between agents. Proposition 2 shows that a crucial determinant for the existence of a steady-state equilibrium is the level of public spending.

The Laffer curve and the Laffer threshold. A supplementary constraint on public spending comes from fact that because taxes are distorting, the amount of public funding that can be raised is bounded. Indeed, a higher tax rate comes at the cost of a smaller tax base, which ultimately diminishes tax revenues when the tax rate is too high. So, not all levels of public spending can be financed by distorting taxes. To see this formally, observe that at the steady state, in the equilibrium characterized by equations (31)–(33), the planner budget constraint (30) implies, after some algebra, that the the labor tax τ^L is a solution of the following equation:

$$\tau^L - \frac{1}{1-\alpha}\frac{\frac{G}{Y_{FB}}(1-\tau^L)^{-\varphi} - \overline{g}_1}{1+\frac{1-\beta}{1+\beta}\frac{1}{1+\varphi}+\frac{\varphi}{1+\varphi}} = 0. \tag{39}$$

We study more formally this relationship in Appendix A.6.[7] The existence of a solution to equation (39) depends on the public spending level. When the public spending is low, equation (39) admits two roots, which correspond to the typical Laffer trade-off between tax rate and tax base. The smaller root corresponds to a low tax and a high labor supply, while the higher root corresponds to a high tax and a low labor supply. The planner opts for the lowest tax, which yields the highest welfare. Conversely, if public spending is very high, then equation (39) admits no solution. This reflects that no tax level is able to finance the public spending, which is then not sustainable. A third case corresponds to the limit between sustainability and no sustainability. There is then a unique solution to equation (39), implying that a unique tax rate enables public spending to be financed. We plot the three possibilities in Figure 1.

The third limit case corresponds to the highest public spending that can be financed. Formally, public spending must verify $G \le \overline{g}_{La} Y_{FB}$, where the threshold $\overline{g}_{La}$ is defined as follows:[8]

$$\overline{g}_{La} := \left(\frac{\varphi}{1+\varphi}\right)^{\varphi}\frac{1-\alpha}{1+\varphi}\left(1+\frac{1-\beta}{1+\beta}\frac{1}{1+\varphi}+\frac{\varphi}{1+\varphi}\right)(1+\frac{1}{1-\alpha}\frac{\overline{g}_1}{1+\frac{1-\beta}{1+\beta}\frac{1}{1+\varphi}+\frac{\varphi}{1+\varphi}})^{1+\varphi}. \tag{40}$$

We will henceforth refer to the restriction $G \le \overline{g}_{La} Y_{FB}$ as the Laffer constraint.Whether the Laffer constraint is more stringent than the SW constraint depends on the parameters, and in general both restrictions must be considered.

The steady-state equilibrium existence. We can provide our main result regarding steady-state equilibrium existence in the following proposition.

[7] As the Laffer curve is well known, we derive the Algebra in the Appendix.

[8] Since $\overline{g}_1 \ge -\frac{1-\beta}{1+\beta}\frac{1-\alpha}{1+\varphi}$, $\overline{g}_{La}$ is always well-defined.

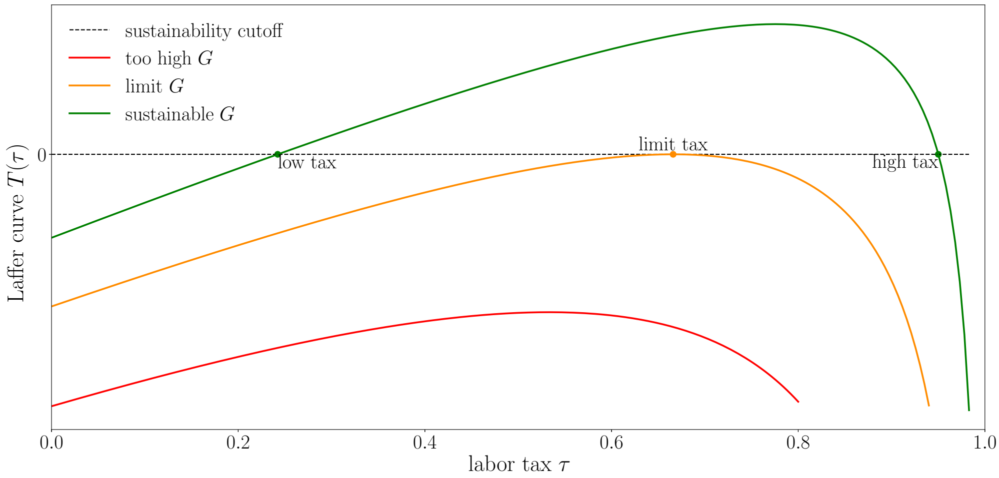

Figure 1: Examples of three Laffer curves for three different values of G. The three cases correspond to: (i) two admissible tax values; (ii) a unique limit tax value; (iii) no admissible tax. The parametrization is: $\beta = 0.97$, $\alpha = 0.3$, $\phi = 0.5$, $\delta = 1.0$, and G/Y_{FB} takes one of the three values in $[0.2, 0.3631, 0.6]$.

Proposition 2 *When $\overline{g}_1 Y_{FB} \leq G$, $G \leq \overline{g}_{SW} Y_{FB}$, and $G \leq \overline{g}_{La} Y_{FB}$, there exists a steady-state equilibrium with binding credit constraint for unemployed agents where both taxes τ^L and τ^K are positive.*

We can verify that $\overline{g}_1 \leq \overline{g}_{SW}$ and $\overline{g}_1 \leq \overline{g}_{La}$. The former inequality is a direct implication of the definition (37) of $\overline{g}_{SW}$, while the latter is proved in Lemma 1 of Appendix A.6. Therefore, the credit-constrained equilibrium always exists for some values of public spending. It can also be observed that when $G = \overline{g}_1 Y_{FB}$, equations (36) and (100) imply $\tau^L = \tau^K = 0$, as in the perfect risk-sharing equilibrium. As a consequence, there is no discontinuity between the first-best and the credit-constraint equilibria around $G = \overline{g}_1 Y_{FB}$.

3.5 The non-existence of the equilibrium with no capital tax

Until now, we have only mentioned two equilibria: the first-best and the one where unemployed agents are credit-constrained and $\tau^K > 0$. We could thus wonder whether a full risk-sharing equilibrium with positive labor tax and hence null capital tax would exist. We can show that such an equilibrium doesn't exist because it is always dominated by an allocation where labor tax is reduced and capital tax is positive. The next proposition summarizes the result.

Proposition 3 *When the equilibrium with $\tau^K > 0$ exists (i.e., conditions of Proposition are fulfilled), then no steady-state equilibrium with $\tau^K = 0$ exists.*

The proof is in Appendix A.7, but the intuition can be simply provided. Imposing $\tau^K = 0$ implies full consumption-smoothing, but also means that public spending should be solely financed out of the labor tax. The distortions implied by this high labor tax involve a small labor supply and a low private consumption. This makes the aggregate welfare lower than an equilibrium where public spending financing relies on both capital and labor taxes. Even though consumption smoothing is not perfect, the latter case means a higher labor supply and a higher private consumption. In other words, for any level of public spending, financing this public spending through a combination of capital and labor taxes generates smaller distortions than financing that relies solely on labor tax. This can be seen in equation (36), and its discussion, which shows that the planner opts for a positive capital tax when the labor tax is positive, and therefore balances allocation efficiency and consumption inequality.

3.6 When is optimal public debt positive?

We now show that this simple model can generate both a positive capital tax and a positive amount of public debt. This result is not obvious: Why would the planner provide more public debt to the market (more liquidity in the sense of Woodford, 1990) and then tax the return on public debt with a positive capital tax? The intuition is that capital tax allows the planner to levy financial resources without increasing the distorting labor tax. However, capital tax distorts saving incentives and agents can save too much compared to the optimal level of capital stock. Public debt thus enables the planner to absorb the excess of savings and reconcile the high savings of private agents and the optimal capital stock.

Formally, the financial market clearing condition implies that, when credit constraints are binding for unemployed agents, the steady-state public debt B can be written as follows:

$$B = \frac{\beta}{1-\beta}\left(-\frac{1-\beta}{1+\beta}\frac{1-\alpha}{1+\varphi}\tau_l - \overline{g}_1\right)(\chi w)^{\varphi}\left(\frac{K}{L}\right)^{\alpha}, \tag{41}$$

which implies that the equilibrium features a positive public debt iff:

$$\tau^L < \frac{1+\varphi}{1-\alpha}\frac{1+\beta}{1-\beta}(-\overline{g}_1). \tag{42}$$

This constraint on τ^L can equivalently be stated as a constraint on the public spending G. This is summarized in the following result.

Result 4. *Steady-state public debt is positive: $B \geq 0$ iff $\overline{g}_1 \leq 0$ and $G \leq \overline{g}_{pos} Y_{FB}$, where:*

$$\overline{g}_{pos} = (-\overline{g}_1)\frac{(1+2\varphi)(1+\beta)}{1-\beta}\left(\frac{(1+2\varphi)(1+\beta)}{1-\alpha}\frac{\alpha}{1+\beta(\delta-1)}\right)^{\varphi}. \tag{43}$$

The proof is in Appendix A.9. Several remarks are in order. First, the threshold $\overline{g}_{\mathrm{pos}}$ depends positively on the capital share. In an economy without capital $\alpha = 0$, $\overline{g}_{\mathrm{pos}} = 0$ and optimal public debt is always negative. It is only when the planner can get some resources out of capital income that optimal public debt can indeed be positive. Second, optimal public debt is positive only when $\overline{g}_1 < 0$, which precludes from Proposition 1 the existence of the first-best equilibrium for the level of public spending G. Third, public debt is positive when public spending is not too high, and thus both labor and capital tax are not too high. Indeed, an increase in public spending leads the planner to increase labor and capital taxes, which are both distorting. This reduces private savings and hence diminishes the room for public debt, for the same targeted pre-tax marginal productivity of capital (35). Fourth and more intuitively, in an equilibrium with positive public debt, the equilibrium saving of employed agents is higher than the optimal capital stock, and the extra savings are absorbed by the public debt. From this allocation, decreasing public debt would inefficiently increase the capital stock, and would require an increase in the capital tax to reduce savings. This would hinder consumption smoothing (see below for an example).

3.7 Dynamic analysis of public debt in the relevant case

We now study the optimal dynamics of public debt after a public spending shock. To simplify the algebra, we focus on the case with full capital depreciation: $\delta = 1$. We denote with a hat the relative deviation to the steady-state value: $\widehat{x}_t = \frac{x_t - x}{x}$ for generic variable x_t with steady-state value x. The public spending shock is assumed to be defined as follows:

$$\widehat{G}_t = \begin{cases} \widehat{G}_0 & \text{if } t = 0, \\ \rho_G \widehat{G}_{t-1} & \text{if } t > 0, \end{cases} \tag{44}$$

with $\widehat{G}_0$ small enough for a first-order approximation of the dynamics to be relevant, and $\rho_G \in (-1, 1)$. The shock only happens at date $t = 0$ and then persists with parameter ρ_G – as is consistent with our assumption of an MIT shock. The process can be written as $\widehat{G}_t = \rho_G^t \widehat{G}_0$. We will analyze the effect of a change in ρ_G, considering two cases. First we analyze the effect of ρ_G with fixed $\widehat{G}_0$ to understand the mechanisms. This experiment has the drawback of comparing shocks with different Net Present Values (NPVs). Our second experiment focuses on studying the effect of ρ_G while keeping the public spending NPV unchanged. More formally we keep the following quantity, denoted by $N\hat{P}V_0$unchanged:

$$N\hat{P}V_0 = \sum_{t=0}^{\infty} \frac{\hat{G}_t}{R^t} = \widehat{G}_0 \sum_{t=0}^{\infty} \left(\frac{\rho_G}{R}\right)^t = \widehat{G}_0 \frac{R}{R - \rho_G}.$$

Keeping the NPV unchanged while changing ρ_G implies setting the initial size of the shock to $\widehat{G}_0(\rho_G) = N\hat{P}V_0 \frac{R-\rho_G}{R}$.

Interestingly, we show in Appendix A.10.2 that the dynamic of the economy can be summarized by the capital as a unique state variable and the public spending shock. It can be computed thanks to a first-order development around the steady-state allocation. The outcome is gathered in the following result.

Result 5. *The dynamic of the capital stock is given by the following system:*

$$\widehat{K}_t = \rho_K \widehat{K}_{t-1} + \sigma_K \widehat{G}_t, \tag{45}$$

where $\rho_K > 0, \sigma_K < 0$, ρ_K doesn't depend on ρ_G and $\frac{\partial \sigma_K}{\partial \rho_G} > 0$.

The expressions of the coefficients are given in Appendix A.10.2. Thus at impact, an increase in public spending diminishes capital, and the higher the persistence of the public spending shock, the stronger the effect.

The dynamic system (45) is stable when the autoregressive coefficient ρ_K is smaller than one in absolute value. In our setup, this is equivalent to verifying Blanchard-Kahn conditions. The result regarding system stability is summarized in the following proposition.

Proposition 4 *The system (45) is stable – $|\rho_K| < 1$ – iff:*

$$\alpha \leq \frac{1}{1 + (1 - \beta)(1 + \varphi)}. \tag{46}$$

The dynamic system is stable under condition (46), which imposes an upper bound on α. Note that this upper bound is always strictly smaller than one and hence can be binding. This condition on α always holds when public debt is positive, i.e., when $\overline{g}_1 < 0$.

By induction, we can derive from (44) and (45) the closed-form expression of the capital IRF,

$$\widehat{K}_t = \sigma_K \widehat{G}_0 \frac{\rho_K^{t+1} - \rho_G^{t+1}}{\rho_K - \rho_G}, \tag{47}$$

which allows us to completely characterize the capital path following a public spending shock. At impact and after a positive shock ($\widehat{G}_0 > 0$), the relative variation of capital is always negative by a quantity $\sigma_K \widehat{G}_0 < 0$. Then, the profile of the capital variation is humped-shaped: it starts decreasing further, before increasing and reverting back to zero.

Role of the persistence of the public spending shock ρ_G on public debt. From the expression of capital (47), it is possible to derive the explicit expression for the optimal dynamics of public debt:

$$\widehat{B}_t = \widehat{G}_0(\Theta^K \rho_K^t - \Theta^G \rho_G^t), \tag{48}$$

where the coefficients Θ^K, Θ^G are functions of the parameters of the model, but not of $\widehat{G}_0$ and are provided in Appendix A.10.2. These parameters can be either positive of negative. On impact, the change in public debt after a positive public spending shock ($\widehat{G}_0 > 0$) can be either positive or negative $\widehat{B}_0 = \widehat{G}_0(\Theta^K - \Theta^G)$, as the sign $\Theta^K - \Theta^G$ is ambiguous. We can characterize the effect of the persistence of the shock on the initial change of public debt:

Proposition 5 *Let assume that the steady-state public debt is positive: $B > 0$. Denoting by $\widehat{B}_0$ the public debt variation on impact, we have:*

$$\left.\frac{\partial \widehat{B}_0}{\partial \rho_G}\right|_{\hat{G}_0} < 0.$$

Moreover, if we further assume $\widehat{B}_0 > 0$, we also have:

$$\left.\frac{\partial \widehat{B}_0}{\partial \rho_G}\right|_{N\hat{P}V_0} < 0.$$

The proof is in Appendix A.10.2. The case of the change in public period-0 debt varying the persistence with fixed initial shock $\hat{G}_0$ ($\left.\frac{\partial \widehat{B}_0}{\partial \rho_G}\right|_{\hat{G}_0} < 0$) is simpler than the case fixing the total net present value $\left.\frac{\partial \widehat{B}_0}{\partial \rho_G}\right|_{N\hat{P}V_0}$. Instead of discussing different cases, we provide a simple example below.

The intuition that the dynamics of the debt depend on the persistence of the shock is the following. After a positive public spending shock, the capital is always falling, but to implement consumption smoothing, the planner doesn't want to decrease private saving (which is used by unemployed agents to consume). As a consequence, when the persistence of the shock is low, the planner increases public debt to provide a store of value to private agents. Then a small increase in future taxes allows one to reduce the public debt. When the persistence is high, this strategy is very costly in terms of welfare, as the fall of the capital stock is persistent, and the planner would have to increase taxes to reduce public debt in periods where capital and output are low. As a consequence, the planner does not increase public debt to avoid having to raise taxes in the future to stabilize the public debt.

We check on a simple numerical example that the result of Proposition 5 still holds when we consider a non-marginal variation in the persistence. Figure 2 plots the illustrative dynamics of the economy and of the instruments of the planner for two shocks with the same NPV but different persistences, the initial size of the shock $\hat{G}_0$ being adjusted. The parameters are $\alpha = 0.3, \beta = 0.7, \varphi = 0.3, \delta = 1, G = 0.01, \chi = 1$, one can check that $\bar{g}_1 Y_{FB} < G$, $G \leq \overline{g}_{SW} Y_{FB}$, and $G < \overline{g}_{La} Y_{FB}$. This economy has an equilibrium capital tax of 6%, labor tax of 3% and a positive public debt of 0.01. The low-persistence economy is $\rho_G = 0.2$ and is the black solid line, and the high-persistence economy is $\rho_G = 0.9$ and is the blue dashed line.

Panel 1 plots the increase in public spending. For the increase to be the same in NPV, it increases by 1% on impact in the case of low persistence and by 0.44% in the case of high persistence. Panel 2 plots the increase in μ, the social value of public liquidity (i.e. the Lagrange multiplier on the government budget constraint). When the persistence is low, the increase is higher on impact, but much less persistent, compared to the high persistence case. Panel 3 plots the capital tax and Panel 4 the labor tax. When the persistence is low, both capital and

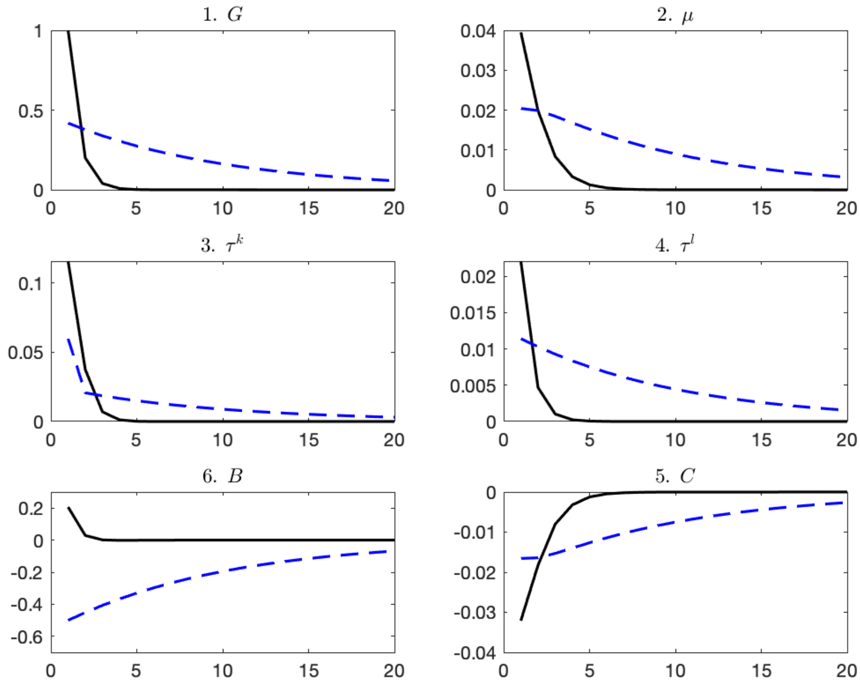

Figure 2: Examples of the dynamics of fiscal variables for a shock with the same Net Present Value and persistences $\rho_G = 0.2$ (black line) and $\rho_G = 0.9$ (blue dashed line) for the parameters, $\alpha = 0.3, \beta = 0.7, \varphi = 0.3, \delta = 1, G = 0.01, \chi = 1$. Variables G, μ, C and B are in proportional change (in %) and τ^K, τ^L are in level change (in %).

labor taxes increase more on impact but are much less persistent. Capital tax increases one order of magnitude more than the labor tax on impact, to front-load the adjustment, as period 0 capital taxes are not distorting (See Farhi, 2010, for similar discussion). However, to avoid reducing the resources of credit-constrained agents, the planner does not fully front-load the adjustment and the labor tax is used on the whole transition. Labor taxes are barely increasing in both economies. As a consequence, there is a persistent increase in both capital and labor tax when the persistence is high, such that any additional increase in taxes (to pay interest on public debt for instance) would be very costly for the planner. This is a strong incentive not to increase public debt. As can be seen in Panel 5, public debt increases in the low-persistence economy, whereas it decreases in the high-persistence economy. Finally, Panel 6 plots aggregate consumption. It falls in both cases, much more when the persistence is low, but returns much faster to its steady-state value.

From this example, we conclude that the dynamics of public debt depend on the persistence of the shock for the same NPV.[9] We now consider a more general model to show that these results do not depend on the structure of this simple model.

4 The general model

4.1 Description

We now solve for the Ramsey allocation, disposing of the simplifying assumptions of Section 3, to consider the general model of Section 2. More precisely, we now assume that i) the period utility function is separable in labor $U(c,l) = u(c) - v(l)$, which is empirically more relevant (see Auclert et al., 2021), ii) there are K idiosyncratic productivity levels, and the transition matrix is a general Markov matrix, iii) the labor tax has an HSV structure $T_t(\tilde{w}yl) := \tilde{w}yl - \kappa_t(\tilde{w}yl)^{1-\tau_t}$, and iv) the Pareto weights may depend on productivity $\omega\left(y_t^i\right)$.

The Ramsey problem consists in choosing the fiscal instruments $(\tau_t^K, \kappa_t, \tau_t, B_t)_{t\geq 0}$ (as a function of the realization of the aggregate shock and of the initial distribution of the state variables of agents) that correspond to the competitive equilibrium with the highest aggregate welfare. Formally, the Ramsey program can be written as follows:

[9]To save some space, we don't present the Figure for the case, with the same initial shock $\hat{G}_0$ and different persistences, as the outcome is the same. It is the same as Figure 2, with a difference scaling.

$$\max_{\left(r_t,w_t,B_t,K_t,L_t,(a_t^i,c_t^i,l_t^i,\nu_t^i)_i\right)_{t\geq 0}} \sum_{t=0}^{\infty} \beta^t \int_i \omega(y_t^i)(u(c_t^i) - v(l_t^i))\ell(di), \tag{49}$$

$$\tag{50}$$

$$G_t + R_t B_{t-1} + (R_t - 1)K_{t-1} + w_t \int_i (y_t^i l_t^i)^{1-\tau_t} \ell(di) = F(K_{t-1}, L_t) + B_t \tag{51}$$

$$\text{for all } i \in \mathcal{I}\text{: } a_t^i + c_t^i = R_t a_{t-1}^i + w_t (y_t^i l_t^i)^{1-\tau_t}, \tag{52}$$

$$a_t^i \geq -\bar{a},\ \nu_t^i(a_t^i + \bar{a}) = 0,\ \nu_t^i \geq 0, \tag{53}$$

$$U_c(c_t^i, l_t^i) = \beta \mathbb{E}_t \left[R_t U_c(c_{t+1}^i, l_{t+1}^i) \right] + \nu_t^i, \tag{54}$$

$$- U_l(c_t^i, l_t^i) = (1 - \tau_t) w_t y_t^i (y_t^i l_t^i)^{-\tau_t} U_c(c_t^i, l_t^i), \tag{55}$$

$$K_t + B_t = \int_i a_t^i \ell(di),\ L_t = \int_i y_t^i l_t^i \ell(di), \tag{56}$$

The Ramsey program consists for the planner to maximize aggregate welfare subject to the governmental budget constraint (50) and to the constraints characterizing the competitive equilibrium: individual budget constraints (8), individual Euler equations (10) and (11), individual credit and positivity constraints (9), market clearing conditions (12) and factor price definitions (1), (4), and (5). We solve this program using a Lagrangian approach, presented in LeGrand and Ragot (2022).[10]

We denote as $\beta^t \lambda_{c,t}^i$ the Lagrange multiplier on the period t Euler equation of agents i, equation (54). When the credit constraint of agents i is binding $a_t^i = -\bar{a}$, and $\lambda_{c,t}^i = 0$, as the Euler equation is not a constraint. It is shown in LeGrand and Ragot (2022) that (when the credit constraint does not bind), the equilibrium can feature either $\lambda_{c,t}^i > 0$ or $\lambda_{c,t}^i < 0$ depending on whether the agents save too much or too little *seen from* the planner's perspective. Similarly, we denote by $\beta^t \lambda_{l,t}^i$ the Lagrange multiplier on the labor supply (55), and by $\beta^t \mu_t$ the Lagrange multiplier on the government budget constraint (50).

To save place, we derive the first-order conditions of the planner in Appendix B. Note that we follow the literature and assume that the solution is interior and first-order conditions of the planner are sufficient to characterize the optimal allocation. We provide some quantitative checks below.

To simplify the interpretation of the first-order conditions of the Ramsey program, we

[10] In LeGrand and Ragot (2022), we show that this method can be used with occasionally binding credit constraints, taking limits of penalty functions. See also Açikgöz et al. (2018) to solve for policies with a utilitarian social welfare function.

introduce the marginal social valuation of liquidity for agent i, defined as:

$$\psi_t^i := \omega_t^i U_c(c_t^i, l_t^i) - \left(\lambda_{c,t}^i - (1+r_t)\lambda_{c,t-1}^i\right) U_{cc}(c_t^i, l_t^i) \\ + \lambda_{l,t}^i \left(U_{cl}(c_t^i, l_t^i) - (1-\tau_t) w_t (y_t^i)^{1-\tau_t} (l_t^i)^{-\tau_t} U_{cc}(c_t^i, l_t^i)\right). \tag{57}$$

This complex expression has a simple interpretation. It is the net value for the planner of transferring one unit of resources to agents i (if it could). First, the gain for the planner would be to increase marginal utility, weighted with the relevant weight $\omega_t^i U_c(c_t^i, l_t^i)$. Second, one additional unit of resources to agent i changes the incentive to save from period $t-1$ to period t, captured by the term with $\lambda_{c,t-1}^i$. Third, this also affects the incentive to save from period t to period $t+1$, captured by the term with $\lambda_{c,t}^i$. Fourth, it affects the incentive to work, captured by the terms in $\lambda_{l,t}^i$. For these last three terms, the effect is multiplied by the marginal change in the marginal utility of consumption, which is the term $U_{cc}(c_t^i, l_t^i)$.

From (57), we also define the net social valuation of liquidity that accounts for the opportunity cost of liquidity, measured by the Lagrange multiplier:

$$\hat{\psi}_t^i := \psi_t^i - \mu_t. \tag{58}$$

With this notation, the first-order conditions of the planner can be easily interpreted. First, for an unconstrained agent i, the planner implements a liquidity smoothing condition:

$$\hat{\psi}_t^i = \beta \mathbb{E}_t R_{t+1} \hat{\psi}_{t+1}^i, \tag{59}$$

where the expectation is taken with respect to the idiosyncratic risk. Equation (59) is a generalized version of the Euler equation (10) (and it is actually the same equation, when all Lagrange multipliers are 0), in which the planner internalizes in the definition of $\hat{\psi}_t^i$ the general equilibrium externalities when setting individual savings.

The first-order condition with respect to labor can be written as:

$$\psi_{l,t}^i = (1-\tau_t) w_t y_t^i (y_t^i l_t^i)^{-\tau_t} \hat{\psi}_t^i \\ + \mu_t F_{L,t} y_t^i - \lambda_{l,t}^i (1-\tau_t) \tau_t w_t y_t^i (y_t^i l_t^i)^{-\tau_t} U_c(c_t^i, l_t^i)/l_t^i, \tag{60}$$

where we have defined:

$$\psi_{l,t}^i := -\omega_t^i U_l(c_t^i, l_t^i) - \lambda_{l,t}^i U_{ll}(c_t^i, l_t^i) \\ + (\lambda_{c,t}^i - R_t \lambda_{c,t-1}^i - \lambda_{l,t}^i (1-\tau_t) w_t (y_t^i)^{1-\tau_t} (l_t^i)^{-\tau_t}) U_{cl}(c_t^i, l_t^i). \tag{61}$$

Similarly to ψ_t^i for consumption, the quantity $\psi_{l,t}^i$ is the social marginal value of labor supply by agent i. The Ramsey first-order condition (60) is a generalized version of the labor Euler equation (11).

The first-order condition with respect to public debt can be written as:

$$\mu_t = \beta(1 + \tilde{r}_{t+1})\mu_{t+1}, \tag{62}$$

without expectation operator thanks to the MIT shock assumption. Equation (62) shows that the planner aims at smoothing the shadow cost of the government budget constraint through time.

The other first-order conditions with respect to R_t, w_t, and τ_t can respectively be written as:

$$0 = \int_j \left(\hat{\psi}_t^j a_{t-1}^j + \lambda_{c,t-1}^j U_c(c_t^j, l_t^j)\right) \ell(dj), \tag{63}$$

$$0 = \int_j (y_t^j l_t^j)^{1-\tau_t} \left(\hat{\psi}_t^j + \lambda_{l,t}^j (1-\tau_t) U_c(c_t^j, l_t^j)/l_t^j\right) \ell(dj), \tag{64}$$

$$\begin{aligned} 0 = &\int_j (y_t^j l_t^j)^{1-\tau_t} \left(\hat{\psi}_t^j + \lambda_{l,t}^j (1-\tau_t) U_c(c_t^j, l_t^j)/l_t^j\right) \ln(y_t^j l_t^j)(dj) \\ &+ \int_j \lambda_{l,t}^j (y_t^j l_t^j)^{1-\tau_t} (U_c(c_t^j, l_t^j)/l_t^j)\ell(dj). \end{aligned} \tag{65}$$

All these equations have a similar interpretation. They involve equalizing the net valuation of liquidity aggregated over the whole population with the relevant weight (e.g., $\int_j \hat{\psi}_t^j a_{t-1}^j \ell(dj)$ in the case of the interest rate) to the general-equilibrium distortion of the instrument (e.g., distortion of savings incentives for the interest rate).

4.2 Consistency of the two approaches

This paper involves two approaches. First, in Section 3, we develop a limited-heterogeneity model that lends itself to analytical computations. This approach enables us to identify new results in a rigorous way. Second, in Sections 4 and 5 below, we consider a general model that allows one to reproduce an empirically relevant heterogeneity. In particular, the approaches rely on different resolution methods. The analytical approach is solved using direct computation and closed-formula. The quantitative approach relies on the Lagrangian and truncation methods of LeGrand and Ragot (2022) (see Section 5.2 for a detailed description). We verify here that the two approaches yield the same results. We proceed in two steps.

First, we check that the application of this Lagrangian approach to the environment of Section 3 delivers the same FOCs as in equations (31)–(33) of the analytical approach. The details of the computations can be found in Appendix B.1.

Second, we verify that the quantitative outcomes of the two approaches are comparable. More precisely, we consider a specification of the quantitative model that is similar to the one of the analytical model: a GHH utility function, a linear labor tax, a two-state productivity process, and a zero credit constraint. This formally corresponds to Assumptions A–D. The only twist we introduce is that we allow for a more general transition matrix than the anti-diagonal matrix of Assumption C. We consider the transition matrix Π_ε defined as:

$$\Pi_\varepsilon = \begin{bmatrix} \varepsilon & 1-\varepsilon \\ 1-\varepsilon & \varepsilon \end{bmatrix}, \tag{66}$$

where $\varepsilon \in [0,1]$. The matrix Π_ε for $\varepsilon = 0$ corresponds to the anti-diagonal case of Assumption C. We will compute the optimal steady-state fiscal policy (capital and labor taxes) as a function of the parameter ε in the general model and compare it to the optimal fiscal policy of the analytical model – and hence which should correspond to the case $\varepsilon = 0$.

However, standard recursive techniques cannot be used to compute the policy in the quantitative approach – even at the steady state. The problem of the planner could be written recursively, but in this case the state space would include the joint distribution of beginning-of-period wealth and Lagrange multipliers on consumption Euler equations (i.e., the joint distribution of $(a^i_{t-1}, \lambda^i_{c,t-1})_i$). Indeed, beginning-of-period wealth a^i_{t-1} and the past value of the Lagrange multiplier λ^i_{t-1} both appear in the first-order conditions of the Ramsey program. To compute the solution, we follow LeGrand and Ragot (2022), and we consider a truncated representation of this problem. We provide the details of the analytical implementation of the truncation approach in Appendix C. Le Grand and Ragot (2022) prove the accuracy of the truncation approach to solve for optimal policies, thus we skip this case here.

We use the same calibration as in Figure 2, namely: $\alpha = 0.3$, $\beta = 0.7$, $\varphi = 0.3$, $\delta = 1$, $G = 0.01$, $\chi = 1$. This calibration guarantees the existence of a positive debt and a positive capital tax in the analytical model (when $\varepsilon = 0$). We compute the optimal steady-state fiscal policy as a function of ε. We plot the results in Figure 3. The first observation is for low values of ε (from 10^{-6} to 10^{-10}): the outcome of the quantitative model is very similar to the one for the analytical model. We have also solved the quantitative model with $\varepsilon = 0$ (but using the Lagrangian and the truncation approach and not the closed-form expressions of the analytical model) and we find very close results. This first finding shows that the quantitative resolution is consistent with the analytical method and that results are continuous in ε. The second observation is when ε increases beyond 10^{-5}, the capital tax diminishes sharply, while the labor tax goes up. This result is consistent with intuition. Indeed, in this very stylized setup, a higher ε means that a higher share of the population remains unemployed with a null income. Their sole resource is their savings. Diminishing the capital tax fosters savings and enables agents to better self-insure against the null income risk. Increasing the labor tax enables the government to balance its budget – since public spending remains fixed.

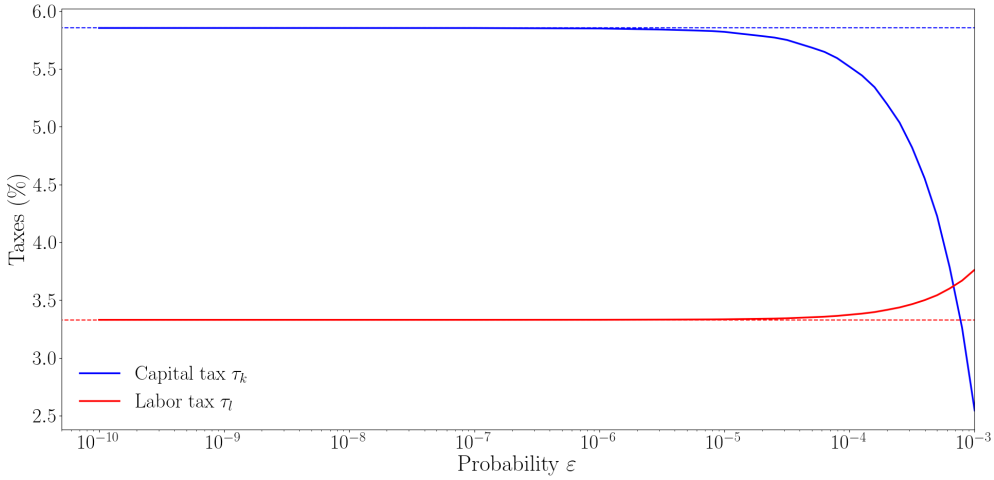

Figure 3: Comparison of the results of the quantitative model (plain lines) to those of the analytical model (dashed lines).

5 Quantitative analysis of the general model

We now show that the intuitions concerning the dynamics of public debt derived in the simple environment are valid when considering a realistic calibration of the general setup. As we are interested in the dynamics of the public debt, and not the optimality of the overall tax system, we use the following strategy. First, we calibrate standard parameters to obtain a realistic steady-state allocation in light of parameters of actual US fiscal policy. Second, following the inverse taxation problem (Bourguignon and Amadeo, 2015; Chang et al., 2018; Heathcote and Tsujiyama, 2021), we estimate an "empirically motivated" social welfare function, such that this steady-state allocation is optimal for the planner. The gain of this methodology is to observe the dynamics of the tax system, considering a quantitatively realistic initial allocation. Starting from this allocation, we implement period-0 shocks on public spending to observe the dynamics of fiscal instruments after the public spending shock.

5.1 Calibration

The period is a quarter.

Preferences. The utility function is separable in labor $U(c, l) = u(c) - v(l)$, where

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma} \text{ and } v(l) = \frac{1}{\chi} \frac{l^{1+\frac{1}{\phi}}}{1+\frac{1}{\phi}}.$$

We set the inverse of intertemporal elasticity of substitution to $\sigma = 2$, which is a standard value used in the literature. For the disutility of labor, we choose $\phi = 0.5$ to match a Frisch elasticity for labor supply of 0.5, which is the value recommended by Chetty et al. (2011) for the intensive margin in heterogeneous-agent models. The scaling parameter is set to $\chi = 0.05$, which implies normalizing the aggregate labor supply to 1/3. Finally, the discount factor is $\beta = 0.99$.

Idiosyncratic risk. We focus on a standard AR(1) process:

$$\log y_t = \rho_y \log y_{t-1} + \varepsilon_t^y,$$
$$\text{where: } \varepsilon_t^y \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma_y^2).$$

Following the strategy of Castaeneda et al. (2003), we choose the parameters (ρ_y, σ_y) to target three key moments.[11] The first target is the variance of the logarithm of consumption, which enables us to capture consumption inequality. Heathcote and Tsujiyama (2021) report a value of $\text{Var}(\log c) = 0.23$. We also target the log-variance of wages to match income inequality, which is found to be $\text{Var}(\log w) = 0.47$ by Heathcote and Tsujiyama (2021). The third target is the debt-to-GDP ratio, which allows us to replicate a realistic financial market equilibrium. We target a value of $B/Y = 61.5\%$, which is the mean ratio over the period (Dyrda and Pedroni, 2018). Calibrating these three moments yields $\rho_y = 0.993$ and $\sigma_y = 0.082$. These parameters are close to those from a direct estimation of the productivity process on PSID data, which corresponds to $\rho_y = 0.9923$ and $\sigma_y = 0.0983$ (see Boppart et al., 2018, and Krueger et al., 2018). The data targets and their model counterparts are reported in Table 1. This simple

	Data	Model
Variance of log consumption Var(log$c$)	0.23	0.20
Variance of log income Var(log$y$)	0.47	0.49
Debt-to-GDP ratio B/Y	61.5%	61.4%

Table 1: Model calibration: targets and model counterparts.

representation is doing a good job in matching the three targeted moments. Furthermore, we can check that this calibration generates a reasonable wealth distribution, even though we do not calibrate it explicitly.[12] Indeed, the calibrated model implies a Gini coefficient of wealth equal to 0.66, which is close, even though below, its empirical counterpart of 0.77. It is known that additional model features must be introduced to match the high wealth inequality in the

[11] More precisely, we minimize the quadratic difference between the model-generated moments and their empirical counterpart, following the Simulated Method of Moments. In the current environment, we see this procedure as a "sophisticated" calibration, rather than an actual SMM – as we equally weight the three moments.

[12] For the problem under consideration, we consider that matching the dispersion of consumption may be more important than the distribution of wealth, which motivates the exclusion of this moment from our calibration strategy.

US, such as heterogeneous discount rates (see Krusell and Smith, 1998), or entrepreneurship (Quadrini, 1999), or stochastic financial returns, which are not considered here.

Finally, we discretize the productivity process using the Rouwenhorst (1995) procedure with seven idiosyncratic states.

Technology. The production function is Cobb-Douglas: $F(K, L) = K^{\alpha}L^{1-\alpha} - \delta K$. The capital share is set to $\alpha = 36\%$ and the depreciation rate to $\delta = 2.5\%$, as in Krueger et al. (2018) among others.

Taxes and government budget constraint. The capital tax is taken from Trabandt and Uhlig (2011), who use the methodology of Mendoza et al. (1994) on public finance data prior to 2008. Their estimation for the US in 2007 (before the financial crisis) yields a capital tax (including both personal and corporate taxes) of $\tau^K = 36\%$. For labor we consider the HSV functional form of equation (2). The progressivity of the labor tax is taken from Heathcote et al. (2017), who report an estimate $\tau = 0.181$. We choose κ to match a public-spending-to-GDP ratio equal to 19%, as in Heathcote and Tsujiyama (2021).

Summary. Table 2 provides a summary of the model parameters.

Parameter	Description	Value
	Preference and technology	
β	Discount factor	0.99
α	Capital share	0.36
δ	Depreciation rate	0.025
$\bar{a}$	Credit limit	0
χ	Scaling param. labor supply	0.05
φ	Frisch elasticity labor supply	0.5
	Shock process	
ρ_y	Autocorrelation idio. income	0.993
σ_y	Standard dev. idio. income	0.082
	Tax system	
τ^K	Capital tax	36%
κ	Sacaling of Labor tax	0.75
τ	Progressivity of tax	0.181

Table 2: Parameter values in the baseline calibration. See text for descriptions and targets.

5.2 Truncation and estimating Pareto weights

We provide a detailed account of the computational implementation of the truncation method in Appendix C, which can be of independent interest as solving such Ramsey problems is not straightforward. More precisely, to investigate the optimal dynamics of the instruments after a shock, we start with providing an exact truncated aggregation of the steady-state model, and we then follow the dynamics of the truncated representation using perturbation methods.

The truncation length is set to $N = 3$, which is shown to provide a good representation of the dynamics. We thus consider $7^3 = 343$ different histories. We have to estimate the weights of the social welfare function, such that the first-order conditions of the planner at the steady state are consistent with the actual US tax system (as described in Section 5.1). However, the problem is in general under-identified, since we have only two constraints (for the capital and labor tax) but seven different weights (one per productivity level). Following Heathcote and Tsujiyama (2021), we introduce productivity weights that depend on the productivity level and define a parametric quadratic representation of weights, as follows:

$$\log \omega_y := \theta_1 \log y + \theta_2 (\log y)^2 .$$

As explained in Appendix C, matching capital and labor tax yields $\theta_1 = 0.95$ and $\theta_2 = 0.66$. In an environment without saving, Heathcote and Tsujiyama (2021) estimate the relationship $\log \omega_y = \theta \log y$ and find a positive value $\theta = 0.517$. The quantitative difference mostly comes from the additional instruments we use.[13]

5.3 Model dynamics

We now simulate the optimal dynamics of the four fiscal tools $(\tau_t^{\kappa}, B_t, \kappa_t, \tau_t)$ after a public spending shock occurring in period $t = 0$. The dynamics of the shock are the same as in equation (44) of the analytical section. After an initial shock in period 0, public spending reverts back to equilibrium at a rate ρ_G.

We first plot the dynamics of the model for two values of the persistence of public spending shocks, with the same NPV. The high one is $\rho_G = 0.97$, which is the annual value used by Farhi (2010) on US data. The low value is $\rho_G = 0.6$, which corresponds to some very specific transitory increase in public spending in the US, such as specific episodes of military build-ups. The initial size of the shock is adjusted for the NPV to be the same. Figure 4 first plots the dynamics of shocks and the instruments. It plots public spending shock G, the Lagrange multiplier μ, both in proportional deviations, the labor level, κ, and the progressivity parameter τ, the capital tax,

[13] We cannot strictly reproduce the specification of Heathcote and Tsujiyama (2021) within our framework, as we need two parameters to match the planner's first-order conditions, because we have more instruments. The correlation between the estimated value $\log \omega_y$ and $\log y$ is 0.68 in our model, which is close to the value of Heathcote and Tsujiyama (2021).

both in level deviations and finally public debt B in proportional deviations. The high-persistence value is in blue dashes. The low-persistence value is in a black solid line.

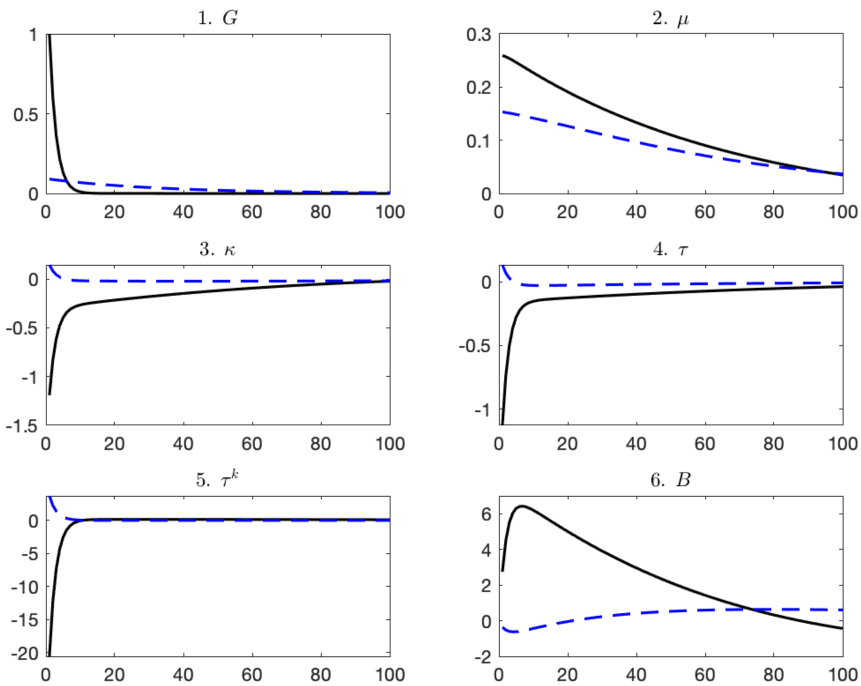

Figure 4: Dynamics of selected variables for two shocks with different persistence and same Net Present Value. Public spending G, value of public resources μ, the level of the tax schedule κ, progressivity of labor tax τ, capital tax τ^k and public debt B. The black solid line is for the persistence $\rho_G = 0.6$. The blue dashed line is for persistence $\rho_G = 0.97$. G, B are in proportional deviations, other variables are in proportional deviations.

All of these variables increase when the persistence is high, but they decrease when persistence is low (which differs from the simple model). Panel 1 represents the dynamics of public spending in proportional deviation. It increases by 1% when $\rho_G = 0.6$ (black solid line), and by 0.09% when $\rho_G = 0.97$ (blue dashed line), for the NPV to be the same. As $G/Y = 19\%$, an increase of 1% of G corresponds to an initial increase of 0.19% of G/Y. Panel 2 plots the value of the Lagrange multiplier (in proportional deviation), which represents the marginal value of additional public resources. The increase in μ is persistently higher in the low persistence case. Panels 3, 4 and 5 report the level κ, tax progressivity and capital tax (in level deviation). In our tax schedule, an increase in κ corresponds to a decrease in the labor tax (as agents receive more labor income). When the persistence is high, the planner reduces labor tax but increases capital tax to levy some resources and decrease public debt (Panel 6). When the persistence is low, the planner increases labor tax and progressivity and public debt increases to finance the high increase in public spending. The difference concerning labor tax between this general model and the simple model (Figure 2 where the labor tax always increases) comes from the assumption of a GHH utility function in the simple model, which is known to bias the fiscal system in favor of labor taxes for public spending shocks. Public debt is represented in Panel 6 in proportional deviation. When persistence is low, public debt actually increases to 6% after four quarters, and then it goes smoothly back to its steady-state value. When persistence is high, public debt decreases on impact and then increases slightly. These non-linear dynamics come from the noticeable tax-smoothing outcome, even in the case of high persistence. When the persistence is high, the autocorrelation of τ, τ^k, and κ are lower than 0.85, much lower than the persistence of G (0.97), whereas the autocorrelation of public debt is high (0.998). It is noticeable that in both cases (high and low persistence) the planner implements a significant change in taxes for a few quarters, and then lets taxes and progressivity converge rapidly to their equilibrium value. Public debt exhibits much more persistent deviations.

Figure 5 plots the dynamics of aggregate variables for the two same economies (the blue dashed line is the high-persistence case, the black solid line is the low-persistence case, both represented in Panel 1 of Fig. 4). It plots output Y, capital K, labor L and total private consumption C, all in proportional deviation.

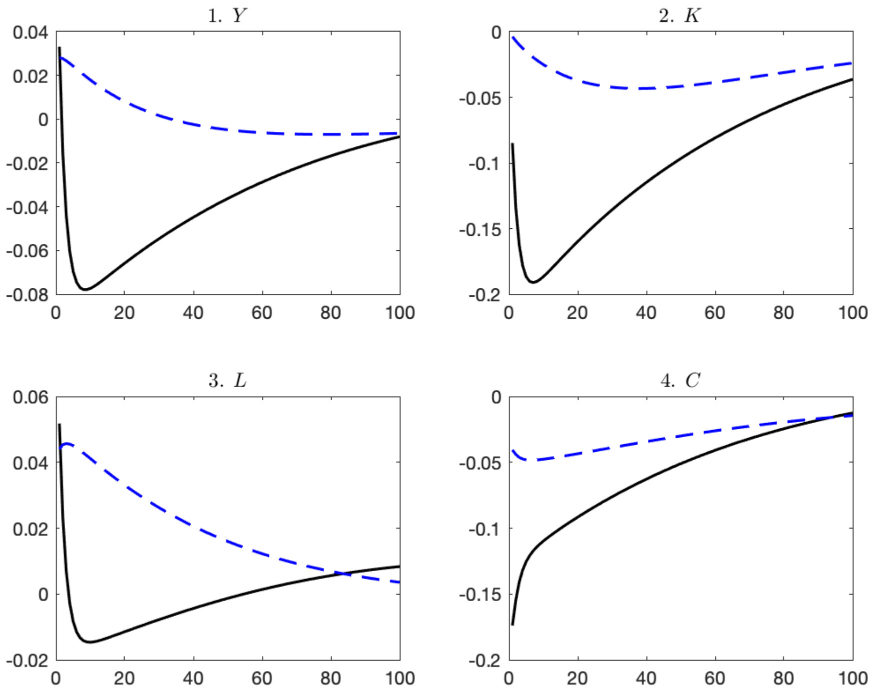

Figure 5: Output, Y, capital K, labour L and consumption. The blue dashed line is for persistence $\rho_G = 0.97$. All variables are in proportional deviations.

First, output increases by roughly the same amount on impact for the two values of persistence. The impact multiplier $\partial Y/\partial G$ is then around 0.2, a value lower than 1, due to the fall in consumption, which can be seen in Panel 4.[14] The capital stock falls in both cases, but much more when the persistence is low. Total labor supply is increasing on impact (due to a negative wealth effect for households). When the persistence is low, the high initial adjustment implies that the initial fall in capital and consumption is higher, compared to the case of low persistence.

To conclude, the properties of the simple model concerning public debt are preserved in the general model. Both the labor and capital taxes have however different dynamics in the general model. Labor tax increases when persistence is low, and decreases when persistence is high. When labor tax increases, progressivity decreases (low persistence) and labor tax decreases when progressivity increases (high persistence). Finally, capital tax decreases on impact when the persistence is low and increases when the persistence is high. As a consequence, although the simple model of Section 3 provides relevant intuitions, the identification of some qualitative aspects of tax changes require a more general model.

5.4 The dynamics of public US spending shock

This section documents the heterogeneity in the persistence of public spending shocks, considering US military spending shocks. We identify these events by large wars, which are, hopefully, in limited numbers. As a consequence, the analysis of this Section involves six well-identified event studies. More precisely, we estimate the persistence of public spending for six military build-ups and then show that the increase in public debt is decreasing with the persistence of the shocks.

We use the data of Ramey and Zubairy (2018) to construct the quarterly time series of public spending, normalized by potential output (see Appendix F for data construction). We consider the six following major military events: World War I (WWI), starting in 1914:Q3, World War II (WWII), starting in 1939:Q3, The Korean War, starting 1950:Q3, the Vietnam War, starting in 1965:Q1, the Soviet Invasion of Afghanistan, starting 1980:Q1, the event of 9/11/2001, in 2001:Q3. As it is obviously hard to identify the expected persistence of all these events, we estimate the persistence of the spending after the peak, when the dynamic of spending is continuously decreasing.

Figure 6 reports the increase of public spending as a percentage of potential output. We also represent the path of the estimated process, which is roughly in line with the data, except for the public spending following 9/11, which is more erratic. The main lesson from Figure 6 is that estimated persistences exhibit sizable differences in their magnitudes. For instance, the persistence for WWI is 0.57, while it amounts to 0.94 for the Vietnam war. Table 3 reports the events in increasing order of the persistence.

For each event, we also compute the change in public debt over the same period, computed

[14] We also computed the cumulative multiplier, which is increasing with persistence.

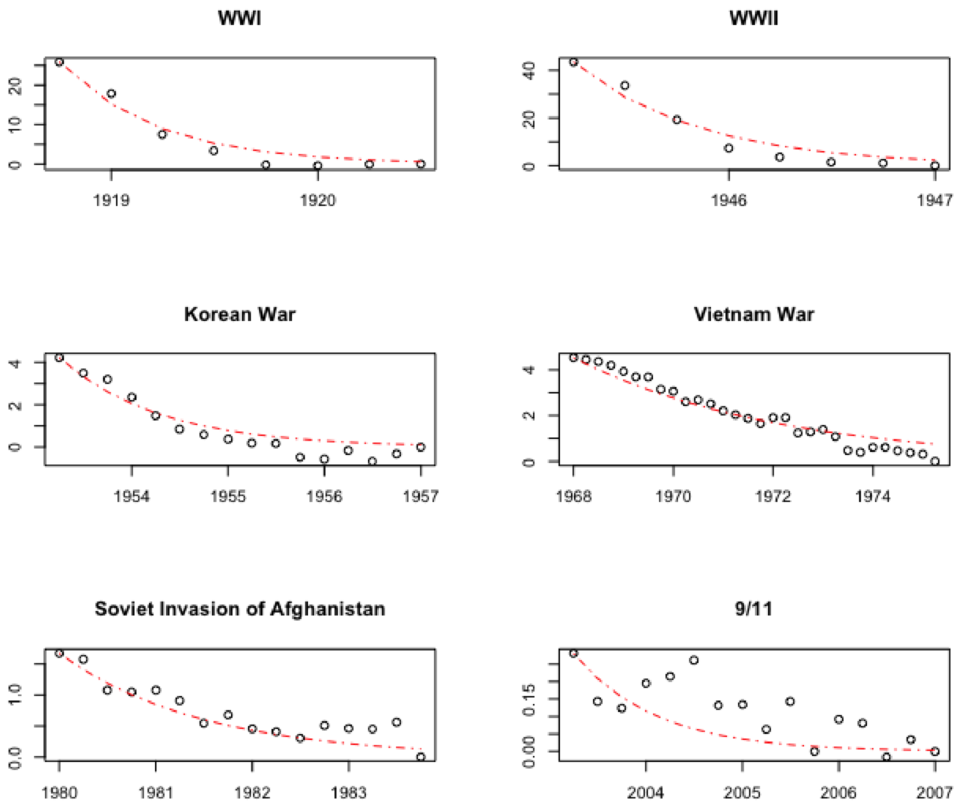

Figure 6: Public spending, deviation from potential output in percent, for six major military events. The black circles are data, the red dashed line is the estimated process, modeled as an AR(1).

as the difference between the maximum and minimum values reached during the period. When public debt is decreasing, this change is negative. To normalize this evolution, we divide this change by the net present value of public spending of the period, discounted with the real market interest rate (see Appendix F). This ratio is reported in percent in the last column, as ΔDebt/G$_{NPV}$. We observe a decreasing trend in the normalized change of public debt as a function of persistence of public spending, which is consistent with the model. Obviously, as we cannot control for relevant factors (such as the conduct of fiscal policy), one should consider these event studies as suggestive evidence only.

6 Conclusion

We investigate the optimal dynamics of the fiscal system after a public spending shock in a heterogeneous-agent model. We first contribute to the clarification of the conditions for relevant

Event	Quart. Pers.(%)	Dates		ΔDebt/G$_{NPV}$(%)
		Beg.	End	
WWI	59	1914:Q3	1920:Q3	7.0
WWII	66	1939:Q3	1947:Q1	6.7
9/11	74	2001:Q3	2007:Q1	1.1
Korean War	78	1950:Q3	1957:Q1	-3.7
Soviet Inv. of Afg.	84	1980:q1	1983:Q4	2.2
Vietnam War	94	1965:Q1	1975:Q2	-1.5

Table 3: Estimated persistence of public spending in percent for the six events, in increasing order and change in public debt divided by the net present value of public spending.

equilibria to exist. The key friction for equilibrium existence is an occasionally-binding credit constraint, which provides a rationale for both positive capital tax and public debt. The second contribution of this paper is to show that the dynamics of public debt and taxes depend crucially on the persistence of the public spending shock. For low persistence, public debt is pro-cyclical, while it is countercyclical for high persistence. In the general model, we find that both capital and labor taxes increase when persistence is high, and decrease otherwise. We consider a quantitative model where the actual US tax system is implemented at the steady state thanks to an inverse optimal taxation approach. The simulation of the quantitative model relies on the Lagrangian-Truncation approach developed in LeGrand and Ragot (2022).

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Appendix

A Property of the simple model

A.1 First-best steady-state allocation in the simple model

We derive the first-best allocation of the simple model. Considering the Utilitarian Social Welfare Function, the Lagrangian associated to the program is simply:

$$\begin{aligned}
\mathcal{L}^{FB} &= \sum_{t=0}^{\infty} \beta^t \left[\log(c_t^u) + \log\left(c_t^e - \chi^{-1} \frac{l_{e,t}^{1+1/\varphi}}{1+1/\varphi} \right) \right] \\
&+ \sum_{t=1}^{\infty} \beta^t \mu_t \left(K_{t-1} + K_{t-1}^{\alpha} l_{e,t}^{1-\alpha} - \delta K_{t-1} - c_t^e - c_t^u - G_t - K_t \right),
\end{aligned}$$

together with non-negativity constraints $c_t^e, c_t^u, l_{e,t} \geq 0$, which are not binding.

To ease the interpretation, we call $L_{FB,t} = l_{e,t}$ the first-best labor supply in this economy. Deriving the FOC yield, after simple manipulations:

$$\frac{1}{c_t^u} = \frac{1}{c_t^e - \chi^{-1} \frac{l_{e,t}^{1+1/\varphi}}{1+1/\varphi}} = \mu_t \tag{67}$$

$$l_{e,t}^{1/\varphi} = \chi \left(1-\alpha\right) K_{t-1}^{\alpha} l_{e,t}^{-\alpha} \tag{68}$$

$$\mu_t = \beta \left(\alpha K_{t-1}^{\alpha-1} l_{e,t}^{1-\alpha} + 1 - \delta \right) \mu_{t+1} \tag{69}$$

At the steady state we have the following equations:

$$\begin{aligned}
\frac{K_{FB}}{L_{FB}} &= \left(\frac{\alpha}{\frac{1}{\beta} + \delta - 1} \right)^{\frac{1}{1-\alpha}}, \\
L_{FB} &= \left(\chi \left(1-\alpha\right)\right)^{\varphi} \left(\frac{\alpha}{\frac{1}{\beta} + \delta - 1} \right)^{\frac{\alpha}{1-\alpha}\varphi}, \\
Y_{FB} &= \left(\chi (1-\alpha)\right)^{\varphi} \left(\frac{\alpha}{\frac{1}{\beta} + \delta - 1} \right)^{\frac{\alpha(1+\varphi)}{1-\alpha}}.
\end{aligned}$$

Finally, using $c_t^u = c^e - \chi^{-1} \frac{L_{FB}^{1+1/\varphi} d}{1+1/\varphi}$ (from (67)) and the resource constraint (24), we can derive consumption shares.

A.2 Proof of Proposition 1

The first-best equilibrium is characterized by optimal consumption smoothing and no inefficient distortions. We now analyze the necessary and sufficient conditions for which the first-best allocation can be decentralized. Using the Euler equations (21) and (22) and consumption smoothing $u'\left(c_e - \frac{1}{\chi}\frac{l_e^{1+\frac{1}{\varphi}}}{1+\frac{1}{\varphi}}\right) = u'(c_u)$, one finds:

$$\beta R_{FB} = 1. \tag{70}$$

To get production and allocation efficiency, distorting taxes must be null $\tau^K = \tau^L = 0$, while the government budget constraint (20) implies that the public debt verifies:

$$B_{FB} = -\frac{\beta}{1-\beta}G < 0.$$

The previous condition is necessary but not sufficient to ensure that the first-best allocation can be implemented. Indeed, an additional constraint is that no agent is credit-constrained. We now check this additional condition.

Factor prices definitions (1) with (70) and $L_{FB} = l_e = (\chi w_{FB})^{\varphi}$ yield:

$$\frac{K_{FB}}{L_{FB}} = \left(\frac{\alpha}{\frac{1}{\beta}+\delta-1}\right)^{\frac{1}{1-\alpha}}, \tag{71}$$

from which we easily deduce:

$$w_{FB} = (1-\alpha)\left(\frac{\alpha}{\frac{1}{\beta}+\delta-1}\right)^{\frac{\alpha}{1-\alpha}}, \tag{72}$$

$$Y_{FB} = K_{FB}^{\alpha}L_{FB}^{1-\alpha} = (\chi(1-\alpha))^{\varphi}\left(\frac{\alpha}{\frac{1}{\beta}+\delta-1}\right)^{\frac{\alpha(1+\varphi)}{1-\alpha}}, \tag{73}$$

$$K_{FB} = \left(\frac{\alpha}{\frac{1}{\beta}+\delta-1}\right)^{\frac{1}{1-\alpha}}(\chi w_{FB})^{\varphi}. \tag{74}$$

Furthermore, since agents are unconstrained, Euler equations imply $c_{u,FB} = c_{e,FB} - \frac{1}{\chi}\frac{l_{e,FB}^{1+\frac{1}{\varphi}}}{1+\frac{1}{\varphi}}$, or after substituting by budget constraints: $R_{FB}a_{u,FB} - a_{e,FB} + \frac{w(\chi w)^{\varphi}}{\varphi+1} = R_{FB}a_{e,FB} - a_{u,FB}$. With (70), this yields:

$$a_{e,FB} - a_{u,FB} = \frac{\beta}{1+\beta}\frac{w_{FB}(\chi w_{FB})^{\varphi}}{\varphi+1}, \tag{75}$$

$$a_{u,FB} + a_{e,FB} = K_{FB} - \frac{\beta}{1-\beta}G, \tag{76}$$

where the second equality is the financial market clearing condition. The combination of both previous equations implies:

$$2\frac{1-\beta}{\beta}\frac{a_{u,FB}}{Y_{FB}} = \overline{g}_1 - \frac{G}{Y_{FB}}, \tag{77}$$

$$\text{with: } \overline{g}_1 = \frac{1-\beta}{\beta}\frac{\alpha}{1/\beta+(\delta-1)} - \frac{1-\beta}{1+\beta}\frac{1-\alpha}{\varphi+1}, \tag{78}$$

Due to the the credit constraint $a_{u,FB} \geq 0$, if the first-best equilibrium exists, equation (77) implies that a condition for the first-best existence is $\frac{G}{Y_{FB}} \leq \overline{g}_1$. We can then deduce $a_{e,FB}$ from (75):

$$a_{e,FB} = a_{u,FB} + \frac{\beta}{1+\beta}\frac{w_{FB}(\chi w_{FB})^{\varphi}}{\varphi+1},$$

which verifies $a_{e,FB} \geq a_{u,FB} \geq 0$.

A.3 Constraint qualification

In our problem, even though the objective function is concave, the equality constraints are not linear and the standard Slater (1950) conditions do not apply. However, we can check that the linear independence constraint qualification (LICQ) holds in our problem. This constraint qualification requires the gradients of equality constraints to be linearly independent at the optimum (or equivalently that the gradient is locally surjective). At any date t, two constraints matter for the instruments of date t. These are the constraints at dates t and $t+1$. We can check that their gradient can be written as:

$$\begin{pmatrix} 1 & \varphi(\chi w_t)^{\varphi}\frac{\tilde{w}_t}{w_t} - (\varphi+1)(\chi w_t)^{\varphi} & -\frac{\beta}{1+\beta}\frac{w_{t-1}(\chi w_{t-1})^{\varphi}}{1+\varphi} \\ -\tilde{r}_{t+1}-1 & \frac{\beta}{1+\beta}(\chi w_t)^{\varphi}\tilde{r}_{t+1} - (R_{t+1}-1)\frac{\beta}{1+\beta}\frac{w_t(\chi w_t)^{\varphi}}{1+\varphi} & 0 \end{pmatrix}, \tag{79}$$

which forms a matrix of rank 2. Indeed, looking at the first and third columns of the matrix in (79) makes it clear that a sufficient condition is $(1+\tilde{r}_{t+1})w_{t-1} \neq 0$. This condition must hold at the optimum, since: (i) equation (1) implies $\tilde{r}_{t+1} \geq 0$, and (ii) we must have $w_{t-1} > 0$.

A.4 Second-order conditions

In the program (29)–(30), we can use the constraint (30) to substitute for the expression of R_t. We can further use financial market constraint (28) to express the public debt B_t as a function of capital K_t and post-tax wage w_t. The planner's program (85)–(86) can be equivalently rewritten as a function of K_t and w_t:

$$\max_{(K_t,w_t)_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\log\left(w_t(\chi w_t)^{\varphi}\right) + \log(K_{t-1} + F(K_{t-1},(\chi w_t)^{\varphi}) \right.$$
$$\left. + \frac{\beta}{1+\beta}\frac{w_t(\chi w_t)^{\varphi}}{1+\varphi} - K_t - G_t - w_t(\chi w_t)^{\varphi}\right).$$

We can further modify this program by defining $W_t = w_t(\chi w_t)^\varphi$ and dropping constants:

$$\max_{(K_t,w_t)_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Big(\log(W_t) \tag{80}$$

$$+ \log\left(K_{t-1} + F(K_{t-1}, \chi^{\frac{\varphi}{1+\varphi}} W_t^{\frac{\varphi}{1+\varphi}}) - \frac{1+\varphi+\varphi\beta}{(1+\beta)(1+\varphi)} W_t - K_t - G_t \right)\Big). \tag{81}$$

The function $(W_t, K_{t-1}) \mapsto F(K_{t-1}, \chi^{\frac{\varphi}{1+\varphi}} W_t^{\frac{\varphi}{1+\varphi}})$ is concave as the composition of concave and increasing functions. We thus deduce that the mapping defined by $(W_t, K_{t-1}, K_t) \mapsto \log(W_t) + \log\left(K_{t-1} + F(K_{t-1}, \chi^{\frac{\varphi}{1+\varphi}} W_t^{\frac{\varphi}{1+\varphi}}) - \frac{1+\varphi+\varphi\beta}{(1+\beta)(1+\varphi)} W_t - K_t - G_t \right)$ is concave. Any interior optimum characterized by first-order conditions must be a maximum.

A.5 FOCs derivation

We focus on the case where unemployed agents are credit-constrained. Note that the situation where both unemployed agents are credit-constrained is not optimal whenever $u'(0) = \infty$. Indeed, when both agents are credit-constrained, deviating and having employed agents to save a small amount yields a finite increase in unemployed agents utility.

Using individual budget constraints, Euler equations (21) and (22) become:

$$u'\left(\frac{w_t(\chi w_t)^\varphi}{\varphi+1} - a_{e,t} \right) = \beta \mathbb{E}_t \left[R_{t+1} u'(R_{t+1} a_{e,t}) \right], \tag{82}$$

$$u'(R_t a_{e,t-1}) > \beta \mathbb{E}_t \left[R_{t+1} u' \left(\frac{w_{t+1}(\chi w_{t+1})^\varphi}{\varphi+1} - a_{e,t+1} \right) \right].$$

Using log preferences, we deduce from Euler equation (82):

$$a_{e,t} = \frac{\beta}{1+\beta} \frac{w_t(\chi w_t)^\varphi}{1+\varphi} \geq 0. \tag{83}$$

After some simplification, the Ramsey program can then be written as:

$$\max_{\{B_t,w_t,R_t\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\log\left(\frac{1}{1+\beta} \frac{w_t(\chi w_t)^\varphi}{\varphi+1} \right) + \log(R_t \frac{\beta}{1+\beta} \frac{w_{t-1}(\chi w_{t-1})^\varphi}{1+\varphi}) \right), \tag{84}$$

$$w_{t+1}(\chi w_{t+1})^\varphi > \beta^2 R_{t+1} R_t w_t (\chi w_t)^\varphi, \tag{85}$$

$$G + B_{t-1} + (R_t - 1) \frac{\beta}{1+\beta} \frac{w_{t-1}(\chi w_{t-1})^\varphi}{1+\varphi} + w_t(\chi w_t)^\varphi = B_t \tag{86}$$

$$+ F(\frac{\beta}{1+\beta} \frac{w_{t-1}(\chi w_{t-1})^\varphi}{1+\varphi} - B_{t-1}, (\chi w_t)^\varphi).$$

Note that the Euler inequality for unemployed agents (85) is equivalent at the steady state to $\beta R < 1$, which will always hold in equilibrium.

The Lagrangian associated to program (84)–(86) can be written (up to some constants independent of policies):

$$\mathcal{L} = (1+\beta)(\varphi+1)\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\log(w_t) + \mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\log(R_t) + \log(a_{e,-1}) \tag{87}$$

$$+\mathbb{E}_0\sum_{t=1}^{\infty}\beta^t\mu_t\Big(F(\frac{\beta}{1+\beta}\frac{w_{t-1}(\chi w_{t-1})^{\varphi}}{1+\varphi} - B_{t-1}, (\chi w_t)^{\varphi}) + B_t - G_t - B_{t-1}$$

$$-(R_t-1)\frac{\beta}{1+\beta}\frac{w_{t-1}(\chi w_{t-1})^{\varphi}}{1+\varphi} - w_t(\chi w_t)^{\varphi}\Big) \tag{88}$$

$$+\mu_0\left(F(K_{-1},(\chi w_0)^{\varphi}) + B_0 - G_0 - B_{-1} - (R_0-1)a_{-1} - w_0(\chi w_0)^{\varphi}\right).$$

Defining by convention w_{-1} as $\frac{\beta}{1+\beta}\frac{w_{-1}(\chi w_{-1})^{\varphi}}{1+\varphi} = a_{-1}$, FOCs associated to the Lagrangian (87) can be summarized as (for $t \geq 0$):

$$0 = (1+\beta)(\varphi+1)\frac{1}{w_t} + \beta(\chi w_t)^{\varphi}\frac{\beta}{1+\beta}\mathbb{E}_t\left[\mu_{t+1}(F_{K,t+1} - R_{t+1} + 1)\right] \tag{89}$$

$$+\chi\mu_t(\chi w_t)^{\varphi-1}\left(\varphi F_{L,t} - (\varphi+1)w_t\right),$$

$$\mu_t = \beta\mathbb{E}_t\left[(1+F_{K,t+1})\mu_{t+1}\right], \tag{90}$$

$$1 = R_t\mu_t\frac{\beta}{1+\beta}\frac{w_{t-1}(\chi w_{t-1})^{\varphi}}{1+\varphi}. \tag{91}$$

We can take advantage of FOCs (90) and (91) to simplify FOC (89) as follows:

$$\mu_t w_t(\chi w_t)^{\varphi}\left(1 - (1+\beta)\varphi\frac{\tau_t^L}{1-\tau_t^L}\right) = (1+\varphi)(1+\beta), \tag{92}$$

which is a time-t equation only and does not raise convergence issues. The only dynamic FOC is the forward-looking equation (90). We will check that the system is well-defined and does not raise convergence issues.

A.6 Steady state and the Laffer Curve

Note that because of FOC (91), $\mu = 0$ or $R = 0$ is not possible at the steady state. FOCs (89)–(91) and governmental budget constraint (86) become at the steady state, where we denote variables without subscripts:

$$\frac{1}{1+\beta}\mu w(\chi w)^{\varphi} = \varphi + 1 + \mu(\chi w)^{\varphi}\varphi(F_L - w), \tag{93}$$

$$1 = \beta(1+F_K) \tag{94}$$

$$1 = R\mu\frac{\beta}{1+\beta}\frac{w(\chi w)^{\varphi}}{1+\varphi} \tag{95}$$

$$F(\frac{\beta}{1+\beta}\frac{w(\chi w)^{\varphi}}{1+\varphi} - B, (\chi w)^{\varphi}) = G + (R-1)\frac{\beta}{1+\beta}\frac{w(\chi w)^{\varphi}}{1+\varphi} + w(\chi w)^{\varphi}. \tag{96}$$

Using (95) and $w = (1 - \tau^L)F_L$, equation (93) becomes:

$$\frac{1}{\beta} - R = \varphi \frac{1+\beta}{\beta}\left(\frac{F_L}{w} - 1\right). \tag{97}$$

Using $w = (1 - \tau^L)F_L$, and $R - 1 = (1 - \tau^K)F_K = (1 - \tau^K)(\beta^{-1} - 1)$, (97) yields:

$$\tau^K = \varphi \frac{1+\beta}{1-\beta}\frac{\tau^L}{1-\tau^L}. \tag{98}$$

After several manipulations and using (94) and (98), as well as the properties of F, the governmental budget constraint (96) implies that τ^L is a solution of the following equation:

$$\tau^L = \frac{1}{1-\alpha}\frac{\frac{G}{Y_{FB}}(1-\tau^L)^{-\varphi} - \overline{g}_1}{1 + \frac{1-\beta}{1+\beta}\frac{1}{1+\varphi} + \frac{\varphi}{1+\varphi}}, \tag{99}$$

where $\overline{g}_1$ is defined in (78). Equation (99) can admit zero, one, or two solutions (as the right hand-side is convex). We now show it more formally: τ^L is thus a solution of the following equation:

$$\mathcal{T}(\tau^L) = 0, \tag{100}$$

$$\text{where: } \mathcal{T} : \tau \in (-\infty, 1) \mapsto \tau - \frac{1}{1-\alpha}\frac{\frac{G}{Y_{FB}}(1-\tau)^{-\varphi} - \overline{g}_1}{1 + \frac{1-\beta}{1+\beta}\frac{1}{1+\varphi} + \frac{\varphi}{1+\varphi}}. \tag{101}$$

The mapping $\tau \mapsto \mathcal{T}(\tau)$ is akin to a Laffer curve. Indeed, we can check that $\mathcal{T}$ is continuously differentiable, strictly concave, with a unique maximum over $(-\infty, 1)$. In consequence, the function $\mathcal{T}$ admits either zero, one, or two solutions. The number of solutions depends on the level of public spending G in (101). When public spending is too high, there is no level of labor tax that makes this public spending sustainable: $T(\tau) < 0$ for all $\tau \in (-\infty, 1)$. When the public spending is sustainable, T typically admits two roots. The smaller root corresponds to a low tax and a high labor supply, while the larger root corresponds to a high tax and a low labor supply. There is a third case that is the limit between sustainability and no sustainability. In this situation, there is a unique tax rate that enables public spending to be financed.

The limit case of the Laffer curve happens when the extremum point of the Laffer curve is the only root of the function. It can be checked that this corresponds to the tax level $\overline{\tau}^L_{La}$ that verifies $\mathcal{T}(\overline{\tau}^L_{La}) = \mathcal{T}'(\overline{\tau}^L_{La}) = 0$, or equivalently to:

$$\tau = \frac{1}{1-\alpha} \frac{\frac{G}{Y_{FB}}(1-\tau)^{-\varphi} - \overline{g}_1}{1 + \frac{1-\beta}{1+\beta}\frac{1}{1+\varphi} + \frac{\varphi}{1+\varphi}}$$

$$\varphi\tau = 1 - \tau - \frac{\varphi}{1-\alpha} \frac{\overline{g}_1}{1 + \frac{1-\beta}{1+\beta}\frac{1}{1+\varphi} + \frac{\varphi}{1+\varphi}}$$

$$\frac{1-\tau}{\varphi} = \frac{1}{1-\alpha} \frac{\frac{G}{Y_{FB}}(1-\tau)^{-\varphi}}{1 + \frac{1-\beta}{1+\beta}\frac{1}{1+\varphi} + \frac{\varphi}{1+\varphi}}$$

$$\overline{\tau}^L_{La} = \frac{1}{1+\varphi} - \frac{1}{1-\alpha}\frac{\varphi}{1+\varphi} \frac{\overline{g}_1}{1 + \frac{1-\beta}{1+\beta}\frac{1}{1+\varphi} + \frac{\varphi}{1+\varphi}}. \tag{102}$$

This corresponds to a ratio of public spending $\frac{G}{Y_{FB}}$, defined as:

$$\overline{g}_{La} := \frac{1-\alpha}{\varphi}\left(1 + \frac{1-\beta}{1+\beta}\frac{1}{1+\varphi} + \frac{\varphi}{1+\varphi}\right)(1-\overline{\tau}^L_{La})^{1+\varphi}, \tag{103}$$

or after some manipulation:

$$\overline{g}_{La} = \left(\frac{\varphi}{1+\varphi}\right)^{\varphi} \frac{1-\alpha}{1+\varphi}\left(1 + \frac{1-\beta}{1+\beta}\frac{1}{1+\varphi} + \frac{\varphi}{1+\varphi}\right)(1 + \frac{1}{1-\alpha}\frac{\overline{g}_1}{1 + \frac{1-\beta}{1+\beta}\frac{1}{1+\varphi} + \frac{\varphi}{1+\varphi}})^{1+\varphi},$$

which is always well-defined since $\overline{g}_1 \geq -\frac{1-\beta}{1+\beta}\frac{1-\alpha}{\varphi+1}$ by definition. So, any public spending such that $\frac{G}{Y_{FB}} > \overline{g}_{La}$ is not sustainable and cannot be financed by any tax system.

Regarding the allocation, we have:

$$c_e = \frac{1}{1+\beta}\frac{w(\chi w)^{\varphi}}{1+\varphi}, \tag{104}$$

$$c_u = \frac{1-(1-\beta)\tau^K}{1+\beta}\frac{w(\chi w)^{\varphi}}{1+\varphi}. \tag{105}$$

Finally, a condition for the $\tau^K > 0$-equilibrium to exist is $c_u > 0$, or equivalently, using (98) which the solution of (99) must verify:

$$(1 + (1+\beta)\varphi)\tau^L < 1. \tag{106}$$

Oppositely, when $\frac{G}{Y_{FB}} < \overline{g}_{La}$, two different tax levels enable the government to finance public spending, and the planner will always opt for the lowest tax rate. Indeed, taxes have an unambiguously negative impact on consumption levels, since they can be written as:

$$c_e = \frac{1}{1+\beta}(1-\tau^L)^{\varphi+1}\frac{w_{FB}(\chi w_{FB})^{\varphi}}{1+\varphi},\ c_u = (1-(1-\beta)\tau^K)c_e. \tag{107}$$

So larger taxes decrease consumption and hence individual welfare.

As a conclusion, let us prove that $\overline{g}_{La} \geq \overline{g}_1$. More formally, we prove the following lemma.

Lemma 1 *We have $\overline{g}_{La} \geq \overline{g}_1$. The equality only holds if $\frac{\varphi}{1-\alpha}\frac{\overline{g}_1}{1+\frac{1-\beta}{1+\beta}\frac{1}{1+\varphi}+\frac{\varphi}{1+\varphi}} = 1$. Otherwise, the inequality is strict.*

Proof. Note that by construction, $\overline{g}_{La} \geq 0$. The result thus holds if $\overline{g}_1 < 0$. We assume that $\overline{g}_1 \geq 0$. Using the definitions of $\overline{g}_{La}$ and $\overline{g}_1$, we have:

$$\frac{\overline{g}_{La} - \overline{g}_1}{\kappa} = \left(\frac{\varphi}{1+\varphi}\right)^{\varphi} \frac{1-\alpha}{1+\varphi}(1+\frac{\overline{g}_1}{\kappa})^{1+\varphi} - \frac{\overline{g}_1}{\kappa},$$
$$\text{where: } \kappa = (1-\alpha)\left(1+\frac{1-\beta}{1+\beta}\frac{1}{1+\varphi}+\frac{\varphi}{1+\varphi}\right) > 0.$$

The sign of $\overline{g}_{La} - \overline{g}_1$ can be determined by focusing on the function $s : x \in \mathbb{R}_+ \mapsto \left(\frac{\varphi}{1+\varphi}\right)^{\varphi}\frac{1}{1+\varphi}(1+x)^{1+\varphi} - x$, which is well-defined and continuously differentiable on $\mathbb{R}_+$. We have $s'(x) \geq 0$ iff $\left(\frac{\varphi}{1+\varphi}\right)^{\varphi}(1+x)^{\varphi} \geq 1$ or $x \geq \varphi^{-1}$. The function s thus admits a minimum for $x = \varphi^{-1}$, whose value is: $s(\varphi^{-1}) = \left(\frac{\varphi}{1+\varphi}\right)^{\varphi}\frac{1}{1+\varphi}(\frac{1+\varphi}{\varphi})^{1+\varphi} - \frac{1}{\varphi} = 0$. We deduce that $s(x) \geq 0$ and the equality holds iff $x = \varphi^{-1}$, which concludes the proof. ■

A.7 Non-existence of the $\tau^K = 0$-equilibrium

We prove here that the steady-state equilibrium featuring full risk-sharing and $\tau^K = 0$ does not exist. More precisely, we show that it is always dominated by the equilibrium featuring binding credit constraint and $\tau^K > 0$ (Sections A.5). We write with the 0-subscript the allocation where $\tau^K = 0$, and with no subscript the allocation where $\tau^K > 0$. The proof is split into two parts: (i) when the $\tau_k > 0$-equilibrium exists, i.e., when condition (106) holds (Section A.7.2); and (ii) when the $\tau_k > 0$-equilibrium does not exist, i.e., when condition (106) does not hold.

A.7.1 Characterization of the $\tau^K = 0$-equilibrium

We focus on the full-insurance equilibrium with zero capital tax. As we use a 0 subscript to denote quantities in this case, we have $\tau_0^K = 0$. With the same steps as in Section A.2, we have:

$$w_0 = (1-\tau^L)w_{FB}, \tag{108}$$
$$K_0 = (1-\tau^L)^{\varphi}K_{FB}, \tag{109}$$
$$Y_0 = (1-\tau^L)^{\varphi}Y_{FB}. \tag{110}$$

Governmental budget constraint (20) becomes:

$$B_0 = -\frac{\beta}{1-\beta}G + \frac{\beta}{1-\beta}\tau_0^L(1-\tau_0^L)^{\varphi}w_{FB}(\chi w_{FB})^{\varphi}.$$

Perfect risk sharing (i.e., $c_{u,0} = c_{e,0} - \frac{1}{\chi}\frac{l_{e,0}^{1+\frac{1}{\varphi}}}{1+\frac{1}{\varphi}}$) and financial market clearing (i.e., $A_0 = K_0 + B_0$) imply (as in (75)), after proper substitution:

$$a_{e,0} - a_{u,0} = \frac{\beta}{1+\beta}\frac{w_{FB}(\chi w_{FB})^{\varphi}}{\varphi+1}(1-\tau_0^L)^{\varphi+1}, \tag{111}$$

$$a_{u,0} + a_{e,0} = (1-\tau^L)^{\varphi}K_{FB} - \frac{\beta}{1-\beta}G + \frac{\beta}{1-\beta}\tau_0^L(1-\tau_0^L)^{\varphi}w_{FB}(\chi w_{FB})^{\varphi}. \tag{112}$$

We deduce by combination of the two previous equations:

$$2a_{u,0} = (1-\tau_0^L)^{\varphi}K_{FB} - \frac{\beta}{1-\beta}G - \frac{\beta}{1+\beta}\frac{w_{FB}(\chi w_{FB})^{\varphi}}{\varphi+1}(1-\tau_0^L)^{\varphi+1}$$
$$+ \frac{\beta}{1-\beta}\tau_0^L(1-\tau_0^L)^{\varphi}w_{FB}(\chi w_{FB})^{\varphi}.$$

Dividing by Y_0 of (110) and using notation (72)–(74) and (26), we obtain:

$$2\frac{a_{u,0}}{Y_0} = \frac{\beta}{1-\beta}(\overline{g}_1 - g_{FB}(1-\tau_0^L)^{-\varphi}) + \left(\frac{1}{1-\beta} + \frac{1}{1+\beta}\frac{1}{\varphi+1}\right)\beta\tau_0^L(1-\alpha). \tag{113}$$

We turn to the computation of $a_{e,0}$. Using (111) and (112), we get:

$$2\frac{a_{e,0}}{Y} = 2\frac{a_{u,0}}{Y} + 2\frac{\beta}{1+\beta}\frac{1-\alpha}{\varphi+1}(1-\tau_0^L),$$

implying that $a_{e,0} \geq a_{u,0}$ for all values of $\tau_0^L \leq 1$. We compute the consumption level $c_{u,0}$ from individual budget constraint (17):

$$2\frac{c_{u,0}}{Y_{FB}} = (1-\tau_0^L)^{\varphi}\overline{g}_1 - \frac{G}{Y_{FB}} + \frac{2}{1+\beta}\frac{1-\alpha}{\varphi+1}(1-\tau_0^L)^{\varphi} + \frac{\varphi}{\varphi+1}(1-\alpha)\tau_0^L(1-\tau_0^L)^{\varphi}. \tag{114}$$

Computing the derivative of $2\frac{c_{u,0}}{Y_{FB}}$ with respect to the labor tax τ_0^L yields:

$$\frac{1}{\varphi(1-\tau_0^L)^{\varphi-1}}\frac{\partial}{\partial\tau_0^L}2\frac{c_{u,0}}{Y_{FB}} = -\frac{(1-\beta)\alpha}{1+\beta(\delta-1)} - (1-\alpha)\tau_0^L < 0, \tag{115}$$

whenever $\tau_0^L \geq 0$. We deduce from the last inequality that $c_{u,0}$ is decreasing with τ_0^L (and hence aggregate welfare since $c_{u,0} = c_{e,0} - \frac{1}{\chi}\frac{l_0^{1+\frac{1}{\varphi}}}{1+\frac{1}{\varphi}}$). Since $a_{e,0} \geq a_{u,0}$ for all values of τ_0^L, the value of τ_0^L is chosen as small as possible for credit constraints not to bind and hence such that $a_{u,0} = 0$. From (113), τ_0^L is the solution of:

$$\tau_0^L = \frac{1}{1 + \frac{1-\beta}{1+\beta}\frac{1}{\varphi+1}}\frac{g_{FB}(1-\tau_0^L)^{-\varphi} - \overline{g}_1}{1-\alpha}. \tag{116}$$

In words, the planner chooses the lowest possible labor tax to reduce distortions. Finally, regarding allocation, we compute:

$$c_{u,0} = c_{e,0} - \chi^{-1} \frac{l_0^{1+1/\varphi}}{1+1/\varphi} = \frac{1}{1+\beta} \frac{w_0(\chi w_0)^\varphi}{1+\varphi}. \tag{117}$$

Laffer curve. Equation (116) admits 0, 1 or 2 solutions, and reflects some form of Laffer curve. The case with the zero solution appears when no equilibrium exists: the public spending G is too high to be financed and no level of labor tax allows the governmental budget to hold. The case with 2 solutions is the standard case when the equilibrium exists: it features either a low tax/high labor supply or a high tax/low labor supply combination. The planner (since inequality (115) holds) unambiguously opts for the lowest tax. Finally the 1-solution case is a limit case that occurs only for a unique value of public spending.

A.7.2 The program of the planner

The fact that the planner implements $a_{u,0} = 0$ in the equilibrium with full risk-sharing implies that the objective of the planner is actually the same as in the case with binding credit constraints, provided in the program (29). As a consequence, the allocation with $\tau^K = 0$ and $\tau^K > 0$ can be written as the outcome of the same program, with the constraint $\tau^K \geq 0$. More formally, we consider the following program:

$$\max_{\{B_t, w_t, R_t\}} \sum_{t=0}^{\infty} \beta^t \left((1+\beta) \log \left(\frac{1}{1+\beta} \frac{w_t(\chi w_t)^\varphi}{\varphi+1} \right) + \log(\beta R_t) \right) \tag{118}$$

$$G + B_{t-1} + (R_t - 1) \frac{\beta}{1+\beta} \frac{w_{t-1}(\chi w_{t-1})^\varphi}{1+\varphi} + w_t(\chi w_t)^\varphi = B_t \tag{119}$$

$$+ F\left(\frac{\beta}{1+\beta} \frac{w_{t-1}(\chi w_{t-1})^\varphi}{1+\varphi} - B_{t-1}, (\chi w_t)^\varphi\right),$$

$$R_t \geq 1 + \tilde{r}_t \tag{120}$$

where the interest rate $\tilde{r}_t$ in the constraint (120) is taken as exogenous with $\tilde{r}_t = F_{K,t}$. We now show that the previous program has the desired properties.

We start with the case $\tau^K = 0$. Denoting by $\beta^t \mu_t$ the Lagrange multiplier associated to the constraint (119), the maximization with respect to B_t yields: $\mu_t = \beta(1 + F_{K,t+1})\mu_{t+1}$, or at the steady state: $\beta(1 + F_K) = 1$.The constraint (119) implies then at the steady state, using (71)–(74), that the labor tax, denoted $\hat{\tau}_0^l$ verifies:

$$(1-\alpha) \left(1 + \frac{1-\beta}{1+\beta} \frac{1}{1+\varphi} \right) \hat{\tau}_0^l = \frac{g_{FB}}{(1-\hat{\tau}_0^l)^\varphi} - \overline{g}_1, \tag{121}$$

which is the equation as (116) for τ_0^L. Since the planner will also choose the lowest solution for (121), we deduce that $\hat{\tau}_0^l = \tau_0^L$. Consumption levels then mechanically verify equation (117), which proves that the steady-state equilibrium with $\tau^K = 0$ is a steady-state solution of the program (118)–(119) where we impose $\tau_t^K = 0$ at all dates.

We now turn to the unconstrained case ($\tau^K \neq 0$). In that case, the FOCs of the program (118)–(119), with respect to B_t, R_t, and w_t, respectively, are:

$$\begin{aligned}
\mu_t &= \mu_{t+1}\beta(1+F_{K,t}), \\
1 &= R_t\mu_t\frac{\beta}{1+\beta}\frac{w_{t-1}(\chi w_{t-1})^{\varphi}}{1+\varphi}, \\
\frac{(1+\beta)(1+\varphi)}{w_t} &= \frac{\mu_t}{w_t}\left((\varphi+1)w_t(\chi w_t)^{\varphi} - \varphi F_{L,t}(\chi w_t)^{\varphi}\right) \\
&+ \frac{\beta\mu_{t+1}}{w_t}(R_{t+1}-1-F_{K,t+1})\frac{\beta}{1+\beta}w_t(\chi w_t)^{\varphi}.
\end{aligned}$$

At the steady-state, we obtain:

$$1 = \beta(1+F_K), \tag{122}$$

$$1 = R\mu\frac{\beta}{1+\beta}\frac{w(\chi w)^{\varphi}}{1+\varphi}, \tag{123}$$

$$\frac{(1+\beta)(1+\varphi)}{\mu(\chi w)^{\varphi}} = (\varphi+1)w - \varphi F_L + \beta(R-1-F_K)\frac{\beta}{1+\beta}w. \tag{124}$$

With (122) and (123), equation (124) yields, after some manipulation, that taxes $\hat{\tau}^k$ and $\hat{\tau}^l$ verify:

$$\hat{\tau}^k = \varphi\frac{1+\beta}{1-\beta}\frac{\hat{\tau}^l}{1-\hat{\tau}^l},$$

which is the same relationship as (98) for τ^K. As we did in the constrained case, the constraint (119) of the program at the steady state yields for $\hat{\tau}^l$ the same definition as equation (99) for τ^L. We deduce that $\hat{\tau}^l = \tau^L$ and $\hat{\tau}^k = \tau^K$, when τ^L satisfies condition (106). Consumption levels (104) and (105) then easily follow. It is also easy to check that $\tau^K, \tau^L > 0$.

We therefore deduce that the allocation with $\tau^K = 0$ is the solution of a constrained program and is hence dominated by the allocation $\tau_k \neq 0$ – whenever the latter exists.[15]

[15] Note that the argument could not be applied right away from the initial program formulation of Section 3 because with $\tau_k \neq 0$, the constraint $a_{u,t} = 0$ was binding – which is not present anymore with the modified program (118)–(119).

A.8 A non-interior steady-state equilibrium

Here we investigate the case when (99) admits a solution, but when this solution does not verify condition (106). We have:

$$\left(1-(1+\varphi(1+\beta))\tau_t^L\right)(1-\tau_t^L)^{\varphi}\mu_t\tilde{w}_t(\chi\tilde{w}_t)^{\varphi} = (1+\beta)(1+\varphi), \tag{125}$$

$$\frac{\mu_{t+1}}{\mu_t} = \frac{1}{\beta(1+F_{K,t+1})}, \tag{126}$$

$$(1+(1-\tau_t^K)F_{K,t})\mu_t(1-\tau_{t-1}^L)^{\varphi+1}\tilde{w}_{t-1}(\chi\tilde{w}_{t-1})^{\varphi} = \frac{(1+\beta)(1+\varphi)}{\beta}. \tag{127}$$

Equation (125) implies that for all t:

$$\tau_t^L \leq \frac{1}{1+\varphi(1+\beta)}.$$

In particular, $\tau^L = \lim_{t\to\infty}\tau_t^L \leq \frac{1}{1+\varphi(1+\beta)}$. From (125), we also understand that there are possibly non-interior steady states, featuring $\lim_t \mu_t = \infty$ or $\lim_t \tilde{w}_t = \infty$.

First case: $\lim w_t = w^* < \infty$**.**

- The case $w^* = 0$ is not possible. Otherwise there are no resources to pay G.
- Assume that $\lim \mu_t = \infty$, then equation (125) implies $\lim_t \tau_t^L = (1+\varphi(1+\beta))^{-1}$. Equation (127) then yields $\lim_t(1+(1-\tau_t^K)F_{K,t}) = \lim_t R_t = 0$.

Second case: $\lim_t w_t = \infty$**.** We thus have $\lim_t \tilde{w}_t = \infty$. We also have from factor price definitions:

$$\chi\tilde{w}_t = \left(\frac{\chi(1-\alpha)}{(1-\tau_t^L)^{\alpha\varphi}}\right)^{\frac{1}{1+\varphi\alpha}} K_{t-1}^{\frac{\alpha}{1+\varphi\alpha}},$$

which yields $\lim K_t = \infty$ and $\lim_t \frac{K_{t-1}}{(\chi w_t)^{\varphi}} = \infty$. We deduce $\lim_t F_{K,t} = -\delta$. We then deduce $\lim_t \mu_t = \infty$, $\lim_t \tau_t^L = (1+\varphi(1+\beta))^{-1}$, and $\lim_t R_t = 0$.

These two non-stationary equilibria feature $\lim_t \mu_t = \infty$ and $\lim_t R_t = 0$.

A.9 Characterization of positive public debt

The financial market clearing condition (19) implies using (83) and the definition of w:

$$B = (\chi w)^{\varphi}\left(\frac{\beta}{1+\beta}\frac{1-\tau^L}{1+\varphi}F_L - \frac{K}{L}\right),$$

which is positive iff: $\frac{\beta}{1+\beta}\frac{1-\tau^L}{1+\varphi} > \frac{1}{F_L}\frac{K}{L}$. Using the definitions of F and $\overline{g}_1$, we can simplify $\frac{1}{F_L}\frac{K}{L}$ and obtain that $B > 0$ iff:

$$\tau^L < -\frac{1+\varphi}{1-\alpha}\frac{1+\beta}{1-\beta}\overline{g}_1. \tag{128}$$

Using the expression (99) of τ^L, we get an equivalent condition to (128):

$$\begin{aligned} g_{FB}(1-\tau^L)^{-\varphi} &< \overline{g}_{\text{pos}}, \\ \text{where: } \overline{g}_{\text{pos}} &= \frac{1+\beta}{1-\beta}(1+2\varphi)(-\overline{g}_1). \end{aligned} \tag{129}$$

A.10 Model dynamics in the presence of aggregate shocks

A.10.1 Model linearization

Defining:

$$\theta = \frac{1}{1+\varphi}\frac{\beta}{1+\beta}, \tag{130}$$

FOCs (89) and (90) and governmental budget constraint (86) become:

$$\mu_t = \beta(1+\alpha K_t^{\alpha-1}\chi^{(1-\alpha)\varphi}w_{t+1}^{(1-\alpha)\varphi} - \delta)\mu_{t+1}, \tag{131}$$

$$0 = 1 - \mu_t w_t(\chi w_t)^{\varphi}(1-\theta) + \frac{\varphi}{1+\varphi}\mu_t(1-\alpha)K_{t-1}^{\alpha}(\chi w_t)^{\varphi(1-\alpha)}, \tag{132}$$

$$K_{t-1}^{\alpha}(\chi w_t)^{\varphi(1-\alpha)} = G_t + K_t - (1-\delta)K_{t-1} + \frac{1}{\mu_t} + (1-\theta)w_t(\chi w_t)^{\varphi}. \tag{133}$$

We deduce R_t from $1 = R_t\mu_t\theta w_{t-1}(\chi w_{t-1})^{\varphi}$ (i.e., FOC (91)) and B_t from $B_t = \theta w_t(\chi w_t)^{\varphi} - K_t$ (i.e., financial market clearing).

We denote by a hat the proportional deviation to the steady-state value. Formally, for a generic variable x: $\widehat{x} = \frac{x_t - x}{x}$. The linearization of equations (131)–(133) yield after some manipulation:

$$\widehat{\mu}_t - E_t\widehat{\mu}_{t+1} = (1-\beta(1-\delta))((\alpha-1)\widehat{K}_t + (1-\alpha)\varphi E_t\widehat{w}_{t+1}), \tag{134}$$

$$0 = -\alpha\widehat{K}_{t-1} + (A-1)\widehat{\mu}_t + ((\varphi+1)(A-1)+1+\varphi\alpha)\widehat{w}_t, \tag{135}$$

$$\begin{aligned} 0 = {} & \frac{G}{Y}\widehat{G}_t + \frac{\alpha}{\frac{1}{\beta}-(1-\delta)}\left(\widehat{K}_t - \beta^{-1}\widehat{K}_{t-1}\right) - (A-1)\varphi\frac{1-\alpha}{1+\varphi}\widehat{\mu}_t \\ & + (A-1)\,\varphi(1-\alpha)\widehat{w}_t, \end{aligned} \tag{136}$$

where τ^L is defined in (99) and where:

$$A := (1+\frac{1}{\varphi(1+\beta)})(1-\tau^L) > 1, \tag{137}$$

where the inequality comes from condition (106) for the existence of the equilibrium.

A.10.2 Public debt spending shock

In the remainder, we will focus on full capital depreciation: $\delta = 1$.

Dynamic system. In that case, we can show that, when setting:

$$r_\mu = \frac{(1+\varphi)(A-1)+1+\alpha\varphi}{(1+\alpha\varphi)A}, \tag{138}$$

$$t_\mu = (1-\alpha)\frac{(1+\varphi)(A-1)+1}{(1+\alpha\varphi)A}, \tag{139}$$

$$r_K = \frac{1-\alpha}{\alpha\beta}(A-1)\frac{\varphi}{1+\varphi}\left(1+\frac{(1+\varphi)(A-1)}{(1+\varphi)(A-1)+1+\varphi\alpha}\right), \tag{140}$$

$$t_K = \frac{1}{\beta}\frac{(1+\varphi\alpha)A}{(1+\varphi)(A-1)+1+\varphi\alpha}, \tag{141}$$

$$s_K = -\frac{G}{\alpha\beta Y}, \tag{142}$$

we obtain from (134)–(136):

$$E_t\left[\widehat{\mu}_{t+1}\right] = r_\mu\widehat{\mu}_t + t_\mu\widehat{K_t}, \tag{143}$$

$$\widehat{K_t} = r_K\widehat{\mu}_t + t_K\widehat{K}_{t-1} + s_K\widehat{G}_t. \tag{144}$$

Since $A > 1$, it can be checked that the coefficients t_K, r_K, t_μ are positive, while $r_\mu > 1$ and $s_K < 0$. Note that all these coefficients are defined at the steady-state and are independent of the values $\widehat{G}_0, \rho_G$ defining the dynamics of the shock $\widehat{G}_t$:

$$\widehat{G}_t = \rho_G^t\widehat{G}_{t-1}. \tag{145}$$

Deriving a simplified dynamic system. We look for coefficients ρ_K, σ_K, ρ_μ, σ_μ, such that, for $t > 1$:

$$\widehat{K_t} = \rho_K\widehat{K}_{t-1} + \sigma_K\widehat{G}_t \tag{146}$$

$$\widehat{\mu}_t = \rho_\mu\widehat{K}_{t-1} + \sigma_\mu\widehat{G}_t. \tag{147}$$

Combining (143)–(144) yields:

$$\begin{aligned} E_t\widehat{K}_{t+1} &= r_\mu(\widehat{K_t} - t_K\widehat{K}_{t-1} - s_K\widehat{G}_t) + r_K t_\mu\widehat{K_t} + t_K\widehat{K_t} + s_K\rho_G\widehat{G}_t \\ &= -r_\mu t_K\widehat{K}_{t-1} - s_K r_\mu\widehat{G}_t + (r_K t_\mu + r_\mu + t_K)\widehat{K_t} + s_K\rho_G\widehat{G}_t \end{aligned}$$

$$E_t\widehat{K}_{t+1} - (t_K + r_\mu + r_K t_\mu)\widehat{K}_t + r_\mu t_K \widehat{K}_{t-1} = (s_K\rho_G - r_\mu s_K)\widehat{G}_t.$$

Using (146), we obtain that ρ_K must solve the following equation:

$$\rho_K^2 - (t_K + r_\mu + r_K t_\mu)\rho_K + r_\mu t_K = 0, \tag{148}$$

whose discriminant is:

$$D = (t_K + r_\mu + r_K t_\mu)^2 - 4 r_\mu t_K. \tag{149}$$

Since $t_K, r_\mu, r_K, t_\mu \geq 0$, we have $D \geq (t_K + r_\mu)^2 - 4r_\mu t_K = (t_K - r_\mu)^2 > 0$, where the strict inequality comes from $t_K = \frac{1}{\beta r_\mu} > 0$. Equation (148) thus admits two distinct roots, which are:

$$\rho_{K,1} = \frac{t_K + r_\mu + r_K t_\mu + \sqrt{D}}{2} \text{ and } \rho_{K,2} = \frac{t_K + r_\mu + r_K t_\mu - \sqrt{D}}{2}. \tag{150}$$

Since $(t_K + r_\mu + r_K t_\mu)^2 > D > 0$, we deduce that $0 < \rho_{K,2} < \rho_{K,1}$. Furthermore, we can check that a necessary and sufficient condition for the equilibrium to be stable is:

$$\alpha \leq \frac{1}{1 + (1-\beta)(1+\varphi)} < 1, \tag{151}$$

where the second inequality comes from $\beta \in (0,1)$. Note that a sufficient condition for the stability is $\overline{g}_1 < 0$ – which is equivalent to $\alpha \leq \frac{1}{1+(1+\beta)(1+\varphi)}$ and hence implies (151).

Let us prove it. The condition $\rho_{K,2} < 1$ is equivalent to $J := t_K + r_\mu + r_K t_\mu - r_\mu t_K - 1 > 0$. Using equations (138)–(141), we can show that:

$$\begin{aligned}
\frac{J}{J_0} &= (\beta(1+\varphi)(A-1) + (1+\alpha\varphi)(\beta - A)) \\
&\quad + \frac{1-\alpha}{\alpha(1+\varphi)}((1+\varphi)(A-1)+1)\,(2(1+\varphi)(A-1) + 1 + \varphi\alpha)\,, \\
\text{where: } J_0 &= \frac{\varphi(1-\alpha)(A-1)}{\beta(1+\alpha\varphi)A((1+\varphi)(A-1)+1+\varphi\alpha)}.
\end{aligned}$$

Since $A > 1$, $J_0 > 0$ and the sign of J is the one of:

$$\begin{gathered}
\beta(1+\varphi)(A-1) + (1+\alpha\varphi)(\beta - 1 - (A-1)) + \\
\frac{1-\alpha}{\alpha(1+\varphi)}((1+\varphi)(A-1)+1)\,(2(1+\varphi)(A-1)+1+\varphi\alpha)\,,
\end{gathered}$$

which can be seen as a quadratic polynomial in $A-1$, which we denote $P(\cdot)$. After some rearrangement, we obtain:

$$
\begin{aligned}
P(A-1) = & \frac{1+\alpha\varphi}{1+\varphi}(-(1-\beta)(1+\varphi)+\frac{1-\alpha}{\alpha})+ \\
& + (A-1)\left(-(1-\beta)(1+\varphi)+\frac{1-\alpha}{\alpha}+2(1+\alpha\varphi)\frac{1-\alpha}{\alpha}\right) \\
& + (A-1)^2\frac{1-\alpha}{\alpha}2(1+\varphi).
\end{aligned}
$$

A necessary condition for $P(A-1)>0$ for all $A>1$ is $P(0)\geq 0$. However, $P(0)\geq 0 \Rightarrow P'(0)>0$ (since $\beta\in(0,1)$). So, since $P''(0)\geq 0$, $P(0)\geq 0$ is a necessary and sufficient condition for $P(A-1)>0$ for $A>1$. The condition $P(0)\geq 0$ is equivalent to condition (151), which concludes the proof regarding equilibrium stability.

Stability and characterization of the system (146)–(147). The Blanchard-Kahn conditions involve checking that $\rho_K<1$.

Since $0<\rho_{K,2}<\rho_{K,1}$ and $\rho_{K,2}\rho_{K,1}=\beta^{-1}>1$, we must have $\rho_{K,1}>1$, which imposes that $\rho_K=\rho_{K,2}$. The stability Blanchard-Kahn condition requires $\rho_{K,2}<1$. Note that in the limit case when the equilibrium does not exist (i.e., condition (106) holds with equality), and which corresponds to $A=1$, it is straightforward to check that $\rho_{K,2}=1$ and that the dynamic system is not stable.

To characterize further the dynamic system (146)–(147), we deduce from (143)–(144) that ρ_μ is connected through ρ_K with:

$$(r_\mu-\rho_K)\rho_\mu=-t_\mu\rho_K. \tag{152}$$

Since $r_\mu>1$, $t_\mu>0$, and $\rho_K\in(0,1)$, we deduce that $\rho_\mu<0$.

Regarding parameters σ_K and σ_μ, we have from (143)–(144):

$$\sigma_K=r_K\sigma_\mu+s_K, \tag{153}$$

$$r_\mu\sigma_\mu=(\rho_\mu-t_\mu)\sigma_K+\sigma_\mu\rho_G. \tag{154}$$

Equation (154) implies:

$$(r_\mu-\rho_G)\sigma_\mu=(\rho_\mu-t_\mu)\sigma_K. \tag{155}$$

Using $r_\mu>1>\rho_G$ and (152) implying that $\rho_\mu-t_\mu=r_\mu\rho_\mu/\rho_K<0$, we deduce that σ_μ and σ_K have opposite signs. Using $r_K>0$ and $s_K<0$ in equation (153), we deduce that $\sigma_\mu>0>\sigma_K$.

The role of the shock persistence ρ_G. Combining (153) and (154) yields:

$$(r_\mu+(t_\mu-\rho_\mu)r_K)\sigma_\mu=(\rho_\mu-t_\mu)s_K+\sigma_\mu\rho_G,$$

which yields, by the implicit function theorem:

$$(r_\mu - \rho_G + (t_\mu - \rho_\mu) r_K) \frac{\partial \sigma_\mu}{\partial \rho_G} = \sigma_\mu,$$

since only σ_μ (and σ_K) depend on ρ_G. Since $r_\mu > 1 > \rho_G$, and $\sigma_\mu, t_\mu, r_K > 0 > \rho_\mu$, we deduce using the previous equation and (153) that:

$$\frac{\partial \sigma_\mu}{\partial \rho_G} > 0 \text{ and } \frac{\partial \sigma_K}{\partial \rho_G} > 0.$$

The previous derivative, and equation (147), imply $\widehat{\mu}_0 = \sigma_\mu \widehat{G}_0$, which implies that for the same initial shock $\widehat{G}_0$, the increase in $\widehat{\mu}_0$ is higher, the higher the persistence:

$$\left.\frac{\partial \widehat{\mu}_0}{\partial \rho_G}\right|_{\widehat{G}_0} > 0 \tag{156}$$

Then, from (135), we have:

$$\widehat{w}_0 = -\frac{A-1}{((\varphi+1)(A-1)+1+\varphi\alpha}\widehat{\mu}_0 \tag{157}$$

which implies $\left.\frac{\partial \widehat{w}_0}{\partial \rho_G}\right|_{\widehat{G}_0} < 0$. Finally, from $\frac{\partial \sigma_K}{\partial \rho_G} > 0$, we deduce $\frac{\partial \widehat{K}_0}{\partial \rho_G} < 0$.

Dynamic of the capital stock. By induction we can then prove that the dynamics (145) and (146) of $\widehat{G}_t$ and $\widehat{K}_t$ can be written as:

$$\widehat{G}_t = \rho_G^t \widehat{G},$$
$$\widehat{K}_t = \sigma_K \frac{\rho_G^{t+1} - \rho_K^{t+1}}{\rho_G - \rho_K} \widehat{G}_0.$$

Let us define:

$$\phi(t) = \begin{cases} \frac{\rho_K^{t+1} - \rho_G^{t+1}}{\rho_K - \rho_G} & \text{if } \rho_K \neq \rho_G, \\ (t+1)\rho_G^t & \text{if } \rho_K = \rho_G, \end{cases}$$

with $\phi(0) = 1$, $\phi(\infty) = 0$, and:

$$(\rho_K - \rho_G)\phi'(t) = \ln(\rho_K)\rho_K^{t+1} - \ln(\rho_G)\rho_G^{t+1}.$$

We have $\phi'(t_m) = 0$ iff:

$$t_m + 1 = \begin{cases} \frac{\ln(-\ln(\rho_K)) - \ln(-\ln(\rho_G))}{\ln(\rho_G) - \ln(\rho_K)} > 0 & \text{if } \rho_K \neq \rho_G, \\ -\frac{1}{\ln(\rho_G)} > 0 & \text{if } \rho_K = \rho_G. \end{cases}$$

It is direct to check that $\phi'(t) > 0$ iff $t < t_m$. The capital response is procyclical (it has the sign of $\widehat{G}_0$). When $\widehat{G}_0 > 0$, capital increases until date t_m before decreasing and converging back to its steady-state value.

We now investigate the impact of ρ_G on t_m. Defining $r_G := -\ln(\rho_G)$ and $r_K := -\ln(\rho_K)$, we obtain:

$$\frac{\partial t_m}{\partial r_G} = \frac{\frac{r_G - r_K}{r_G} - (\ln(r_G) - \ln(r_K))}{(r_G - r_K)^2} \text{ if } \rho_K \neq \rho_G.$$

By the Taylor-Lagrange theorem, there exists $r \in (r_K, r_G)$, such that:

$$\ln(r_K) - \ln(r_G) = \frac{r_K - r_G}{r_G} - \frac{(r_K - r_G)^2}{2r^2},$$

from which we deduce:

$$\frac{\partial t_m}{\partial r_G} = \begin{cases} \frac{-\frac{(r_K - r_G)^2}{2r^2}}{(r_G - r_K)^2} < 0 & \text{if } \rho_K \neq \rho_G, \\ -\frac{1}{r_G^2} < 0 & \text{if } \rho_K = \rho_G. \end{cases}$$

So t_m decreases with r_G and increases with ρ_G: the more persistent ρ_G, the longer the impact of capital dynamics.

We now study the impact of ρ_G on the $\phi(t_m)$, the maximal value of ϕ (which corresponds to the maximal variation of capital stock following the public spending shock).

$$\phi(t_m) = \begin{cases} \frac{e^{-\frac{\ln(r_G) - \ln(r_K)}{r_G - r_K} r_K} - e^{-\frac{\ln(r_G) - \ln(r_K)}{r_G - r_K} r_G}}{e^{-r_K} - e^{-r_G}} & \text{if } \rho_K \neq \rho_G, \\ r_G^{-1} e^{-r_G(r_G^{-1} - 1)} > 0 & \text{if } \rho_K = \rho_G. \end{cases}$$

We focus on the case where $\rho_K \neq \rho_G$ and $\rho_K < 1$. Note that we have:

$$\phi(t_m) \rightarrow_{\rho_G \rightarrow 1} \frac{1}{1 - \rho_K},$$

and

$$\begin{aligned} \phi(t_m) &= \frac{\left(\frac{r_K}{r_G}\right)^{\frac{r_K}{r_G - r_K}} - \left(\frac{r_K}{r_G}\right)^{\frac{r_G}{r_G - r_K}}}{e^{-r_K} - e^{-r_G}} \\ &= \frac{\left(\frac{r_G}{r_K}\right)^{-\frac{1}{r_G/r_K - 1}} - \left(\frac{r_G}{r_K}\right)^{-\frac{1}{r_G/r_K - 1} - 1}}{e^{-r_K}\left(1 - e^{-r_K(r_G/r_K - 1)}\right)}. \end{aligned}$$

We define $x := r_G/r_K - 1$, such that $\frac{r_G}{r_K} = 1 + x$, $\frac{r_K}{r_G - r_K} = \frac{1}{r_G/r_K - 1} = \frac{1}{x}$, and $\frac{r_G}{r_G - r_K} = 1 + \frac{1}{x}$, and we define $f(x) := \phi(t_m) = \frac{(1+x)^{-\frac{1}{x}} - (1+x)^{-\frac{1}{x} - 1}}{1 - e^{-r_K x}}$, such that:

$$(1 + x)^{\frac{1}{x} + 1} f'(x) = \frac{\frac{\ln(1+x)}{x}(1 - e^{-r_K x}) - x r_K e^{-r_K x}}{(1 - e^{-r_K x})^2}.$$

Note that:

$$(1+x)^{\frac{1}{x}+1} f'(x) \sim_{x\to -1} \frac{\ln(1+x)}{e^{r_K}-1},$$

which is negative whenever x is sufficiently close to -1. In other words, f decreases with $x = r_G/r_K - 1$, and hence increases with ρ_G.

Dynamics of public debt. Regarding public debt, the financial market clearing implies that $B_t = \frac{\beta}{1+\beta}\frac{\chi^\varphi}{1+\varphi} w_t^{1+\varphi} - K_t$ and thus that the dynamics are given by:

$$B\widehat{B}_t \quad = \quad \frac{\beta}{1+\beta}\chi^\varphi w^{1+\varphi}\widehat{w}_t - K\widehat{K}_t.$$

At steady state, $B = \frac{\beta}{1+\beta}\frac{\chi^\varphi}{1+\varphi} w^{1+\varphi} - K$. Define

$$\alpha_B \quad := \quad \frac{1}{B}\frac{\beta}{1+\beta}\frac{\chi^\varphi}{1+\varphi} w^{1+\varphi}$$

such that:

$$\widehat{B}_t = \alpha_B \widehat{w}_t - (\alpha_B - 1)\,\widehat{K}_t.$$

Using equations (146), (147) and (32), one finds:

$$\widehat{B}_t \quad = \quad \Theta^K \widehat{G}_0 \rho_K^t - \Theta^G \widehat{G}_0 \rho_G^t,$$

with

$$\begin{aligned}
\Theta^K \quad &:= \quad \left(\alpha_B \frac{\alpha - (A-1)\,\rho_\mu}{(\varphi+1)(A-1)+1+\varphi\alpha} - (\alpha_B - 1)\,\rho_K\right)\frac{\sigma_K}{\rho_K - \rho_G}, \\
\Theta^G \quad &:= \quad \left(\alpha_B \frac{\alpha - (A-1)\,\rho_\mu}{(\varphi+1)(A-1)+1+\varphi\alpha} - (\alpha_B - 1)\,\rho_K\right)\frac{\sigma_K}{\rho_K - \rho_G} \\
&\quad\quad + \alpha_B \frac{A-1}{(\varphi+1)(A-1)+1+\varphi\alpha}\sigma_\mu + (\alpha_B - 1)\,\sigma_K.
\end{aligned}$$

Effect of persistence on impact. At impact ($t = 0$), we have:

$$B\widehat{B}_0 \quad = \quad -\left(\frac{\beta}{1+\beta}\chi^\varphi w^{1+\varphi}\frac{A-1}{(\varphi+1)(A-1)+1+\varphi\alpha}\sigma_\mu(\rho_G) + \sigma_K(\rho_G) K\right)\hat{G}_0(\rho_G). \quad (158)$$

In the previous expression, we have explicitly noted the dependence on ρ_G. Recall that $\frac{\partial \sigma_\mu}{\partial \rho_G} > 0$, $\frac{\partial \sigma_K}{\partial \rho_G} > 0$, and since the $N\hat{P}V_0$ is fixed and $\hat{G}_0$ endogenous, $\left.\frac{\partial \widehat{G}_0}{\partial \rho_G}\right|_{N\hat{P}V} < 0$.

As a consequence, if the public debt is positive at the steady state ($B > 0$ equivalent to $\bar{g}_1 < 0$ – see Section A.9), then for a positive exogenous initial shock, $\hat{G}_0 > 0$, $\frac{\partial \sigma_\mu}{\partial \rho_G} > 0$, $\frac{\partial \sigma_K}{\partial \rho_G} > 0$ imply $\frac{\partial \widehat{B}_0}{\partial \rho_G} < 0$. The higher the shock persistence, the greater the variation of public debt at impact decreases.

$$\left.\frac{\partial \widehat{B}_0}{\partial \rho_G}\right|_{\hat{G}_0} < 0.$$

The case of constant $N\hat{P}V_0$ is as follows.

$$B \left.\frac{\partial \widehat{B}_0}{\partial \rho_G}\right|_{N\hat{P}V_0} = \left.\frac{\partial \widehat{B}_0}{\partial \rho_G}\right|_{\hat{G}_0} + \frac{B\widehat{B}_0}{\hat{G}_0(\rho_G)} \left.\frac{\partial \widehat{G}_0}{\partial \rho_G}\right|_{N\hat{P}V}.$$

If in addition to $B > 0$, we also have $\widehat{B}_0 > 0$, we deduce since $\left.\frac{\partial \widehat{B}_0}{\partial \rho_G}\right|_{\hat{G}_0} < 0$ and $\left.\frac{\partial \widehat{G}_0}{\partial \rho_G}\right|_{N\hat{P}V} < 0$:

$$B \left.\frac{\partial \widehat{B}_0}{\partial \rho_G}\right|_{N\hat{P}V_0} < 0.$$

B First-order conditions of the individual Ramsey program

The Ramsey problem can be written as follows:

$$\max_{\left(r_t, \tilde{w}_t, \tilde{r}_t, \tau_t^K, \tau_t, \kappa_t, B_t, K_t, L_t, (a_t^i, c_t^i, l_t^i, \nu_t^i)_i\right)_{t\geq 0}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \int_i \omega_t^i U(c_t^i, l_t^i)\ell(di)\right], \tag{159}$$

$$\tag{160}$$

$$G_t + R_t B_{t-1} + (R_t - 1)K_{t-1} + w_t \int_i (y_t^i l_t^i)^{1-\tau_t} \ell(di) = K_{t-1}^{\alpha} L_t^{1-\alpha} - \delta K_{t-1} + B_t \tag{161}$$

$$\text{for all } i \in \mathcal{I}\text{: } a_t^i + c_t^i = R_t a_{t-1}^i + w_t (y_t^i l_t^i)^{1-\tau_t}, \tag{162}$$

$$a_t^i \geq -\bar{a},\ \nu_t^i (a_t^i + \bar{a}) = 0,\ \nu_t^i \geq 0, \tag{163}$$

$$U_c(c_t^i, l_t^i) = \beta \mathbb{E}_t \left[R_{t+1} U_c(c_{t+1}^i, l_{t+1}^i)\right] + \nu_t^i, \tag{164}$$

$$-U_l(c_t^i, l_t^i) = (1 - \tau_t) w_t (y_t^i)^{1-\tau_t} (l_t^i)^{-\tau_t} U_c(c_t^i, l_t^i), \tag{165}$$

$$K_t + B_t = \int_i a_t^i \ell(di),\ L_t = \int_i y_t^i l_t^i \ell\,(di)\,. \tag{166}$$

The Lagrangian can be written as:

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \int_i \omega_t^i U(c_t^i, l_t^i) \ell(di) \tag{167}$$

$$- \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \int_i \left(\lambda_{c,t}^i - R_t \lambda_{c,t-1}^i \right) U_c(c_t^i, l_t^i) \ell(di)$$

$$+ \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \int_i \lambda_{l,t}^i \left(U_l(c_t^i, l_t^i) + (1-\tau_t) w_t (y_t^i)^{1-\tau_t} (l_t^i)^{-\tau_t} U_c(c_t^i, l_t^i) \right) \ell(di)$$

$$- \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \mu_t \Big(G_t + (1-\delta) B_{t-1} + (R_t - 1 + \delta) \int_i a_{t-1}^i \ell(di) + w_t \int_i (y_t^i l_t^i)^{1-\tau_t} \ell(di)$$

$$- (\int_i a_{t-1}^i \ell(di) - B_{t-1})^{\alpha} (\int_i y_t^i l_t^i \ell(di))^{1-\alpha} - B_t \Big), \tag{168}$$

where:

$$c_t^i = -a_t^i + R_t a_{t-1}^i + w_t (y_t^i l_t^i)^{1-\tau_t}. \tag{169}$$

FOC with respect to savings choices. Deriving (167) with respect to a_t^i yields:

$$0 = \beta^t \int_j \omega_t^j U_c(c_t^j, l_t^j) \frac{\partial c_t^j}{\partial a_t^i} \ell(dj)$$

$$- \beta^t \int_j \left(\lambda_{c,t}^j - R_t \lambda_{c,t-1}^j \right) U_{cc}(c_t^j, l_t^j) \frac{\partial c_t^j}{\partial a_t^i}$$

$$+ \beta^t \int_j \lambda_{l,t}^j U_{cl}(c_t^j, l_t^j) \frac{\partial c_t^j}{\partial a_t^i} \ell(dj)$$

$$+ \beta^t (1-\tau_t) w_t \int_j \lambda_{l,t}^j (y_t^j)^{1-\tau_t} (l_t^j)^{-\tau_t} U_{cc}(c_t^j, l_t^j) \frac{\partial c_t^j}{\partial a_t^i} \ell(dj)$$

$$+ \beta^{t+1} \mathbb{E}_t \left[\int_j \omega_{t+1}^j U_c(c_{t+1}^j, l_{t+1}^j) \frac{\partial c_{t+1}^j}{\partial a_t^i} \right]$$

$$- \beta^{t+1} \mathbb{E}_t \left[\int_j \left(\lambda_{c,t+1}^j - R_{t+1} \lambda_{c,t}^j \right) U_{cc}(c_{t+1}^j, l_{t+1}^j) \frac{\partial c_{t+1}^j}{\partial a_t^i} \ell(dj) \right]$$

$$+ \beta^t \mathbb{E}_t \left[\int_j \lambda_{l,t+1}^j U_{cl}(c_{t+1}^i, l_{t+1}^i) \frac{\partial c_{t+1}^j}{\partial a_t^i} \ell(dj) \right]$$

$$+ \beta^{t+1} (1-\tau_{t+1}) w_{t+1} \mathbb{E}_t \left[\int_j \lambda_{l,t+1}^j (y_{t+1}^j)^{1-\tau_{t+1}} (l_{t+1}^j)^{-\tau_{t+1}} U_{cc}(c_{t+1}^j, l_{t+1}^j) \frac{\partial c_{t+1}^j}{\partial a_t^i} \ell(dj) \right]$$

$$+ \beta^{t+1} \mathbb{E}_t \left[\mu_{t+1} \left(\alpha K_t^{\alpha-1} L_{t+1}^{1-\alpha} - (r_{t+1} + \delta) \right) \right].$$

We also denote:

$$\psi_t^i = \omega_t^i U_c(c_t^i, l_t^i) + \lambda_{l,t}^i U_{cl}(c_t^i, l_t^i) \\ - \left(\lambda_{c,t}^i - R_t \lambda_{c,t-1}^i - \lambda_{l,t}^i (1-\tau_t) w_t (y_t^i)^{1-\tau_t} (l_t^i)^{-\tau_t}\right) U_{cc}(c_t^i, l_t^i), \tag{170}$$

and get using $\tilde{r}_{t+1} = \alpha K_t^{\alpha-1} L_{t+1}^{1-\alpha} - \delta$:

$$0 = \int_j \psi_t^j \frac{\partial c_t^j}{\partial a_t^i} \ell(dj) + \beta \mathbb{E}_t \left[\int_j \psi_{t+1}^j \frac{\partial c_{t+1}^j}{\partial a_t^i} \right] \\ + \beta \mathbb{E}_t \left[\mu_{t+1} (\tilde{r}_{t+1} - R_{t+1} + 1) \right].$$

Using (169), we obtain $\frac{\partial c_t^j}{\partial a_t^i} = -1_{i=j}$ and $\frac{\partial c_{t+1}^j}{\partial a_t^i} = R_{t+1} 1_{i=j}$, from which we deduce:

$$\psi_t^i = \beta \mathbb{E}_t \left[R_{t+1} \psi_{t+1}^i \right] + \beta \mathbb{E}_t \left[\mu_{t+1} (1 + \tilde{r}_{t+1} - R_{t+1}) \right].$$

FOC with respect to labor supply. Deriving (167) with respect to l_t^i yields:

$$0 = \int_j \psi_t^j \frac{\partial c_t^j}{\partial l_t^i} \ell(dj) - \psi_{l,t}^j \\ - \lambda_{l,t}^i (1-\tau_t) \tau_t w_t (y_t^i)^{1-\tau_t} (l_t^i)^{-\tau_t - 1} U_c(c_t^i, l_t^i) - \mu_t (w_t (1-\tau_t) (y_t^i)^{1-\tau_t} (l_t^i)^{-\tau_t} - F_{L,t} y_t^i),$$

where we have defined:

$$\psi_{l,t}^i = -\omega_t^i U_l(c_t^i, l_t^i) - \lambda_{l,t}^i U_{ll}(c_t^i, l_t^i) + (\lambda_{c,t}^i - R_t \lambda_{c,t-1}^i - \lambda_{l,t}^i (1-\tau_t) w_t (y_t^i)^{1-\tau_t} (l_t^i)^{-\tau_t}) U_{cl}(c_t^i, l_t^i).$$

Using (169), we obtain $\frac{\partial c_t^j}{\partial l_t^i} = (1-\tau_t) w_t (y_t^j)^{1-\tau_t} (l_t^i)^{-\tau_t} 1_{i=j}$, which implies:

$$\psi_{l,t}^i = (1-\tau_t) w_t y_t^i (y_t^i l_t^i)^{-\tau_t} \hat{\psi}_t^i \\ + \mu_t F_{L,t} y_t^i - \lambda_{l,t}^i (1-\tau_t) \tau_t w_t y_t^i (y_t^i l_t^i)^{-\tau_t} U_c(c_t^i, l_t^i) / l_t^i.$$

FOC with respect to the interest rate. Deriving (167) with respect to R_t yields:

$$0 = \int_j \left(\psi_t^j \frac{\partial c_t^j}{\partial R_t} + \lambda_{c,t-1}^j U_c(c_t^j, l_t^j) \right) \ell(dj) - \mu_t \int_j a_{t-1}^j \ell(dj).$$

From (169), we obtain $\frac{\partial c_t^j}{\partial R_t} = a_{t-1}^j$, which yields:

$$0 = \int_j \left(\hat{\psi}_t^j a_{t-1}^j + \lambda_{c,t-1}^j U_c(c_t^j, l_t^j) \right) \ell(dj).$$

FOC with respect to the wage rate. Deriving (167) with respect to w_t yields:

$$0 = \int_j \left(\psi_t^j \frac{\partial c_t^j}{\partial w_t} + \lambda_{l,t}^j (1-\tau_t)(y_t^j)^{1-\tau_t} (l_t^j)^{-\tau_t} U_c(c_t^j, l_t^j) \right) \ell(dj)$$
$$- \mu_t \int_j (y_t^j l_t^j)^{1-\tau_t} \ell(dj).$$

From (169), we get $\frac{\partial c_t^j}{\partial w_t} = (y_t^j l_t^j)^{1-\tau_t}$ and:

$$0 = \int_j (y_t^j l_t^j)^{1-\tau_t} \left(\hat{\psi}_t^j + \lambda_{l,t}^j (1-\tau_t) U_c(c_t^j, l_t^j)/l_t^j \right) \ell(dj).$$

FOC with respect to public debt. Deriving (167) with respect to B_t yields:

$$0 = \mu_t - \beta \left[(1 - \delta - \alpha K_{t-1}^{\alpha} L_t^{1-\alpha} \mu_{t+1} \right],$$

or using the definition of $\tilde{r}_{t+1}$:

$$\mu_t = \beta(1 + \tilde{r}_{t+1}) \mu_{t+1}.$$

FOC with respect to progressivity. Deriving (167) with respect to τ_t yields:

$$0 = \int_j \psi_t^j \frac{\partial c_t^j}{\partial \tau_t} \ell(dj)$$
$$+ w_t \int_j \lambda_{l,t}^j \frac{\partial}{\partial \tau_t} \left((1-\tau_t)(y_t^j l_t^j)^{1-\tau_t} \right) (U_c(c_t^j, l_t^j)/l_t^j) \ell(dj)$$
$$- \mu_t w_t \int_j \frac{\partial}{\partial \tau_t} \left((y_t^j l_t^j)^{1-\tau_t} \right) \ell(dj).$$

From (169), we have $\frac{\partial c_t^j}{\partial \tau_t} = (y_t^j l_t^j)^{1-\tau_t}$ and:

$$0 = \int_j \hat{\psi}_t^j \frac{\partial}{\partial \tau_t} \left((y_t^j l_t^j)^{1-\tau_t} \right) (dj)$$
$$+ \int_j \lambda_{l,t}^j \left(-(y_t^j l_t^j)^{1-\tau_t} + (1-\tau_t) \frac{\partial}{\partial \tau_t} (y_t^j l_t^j)^{1-\tau_t} \right) (U_c(c_t^j, l_t^j)/l_t^j) \ell(dj),$$

and

$$0 = \int_j \left(\hat{\psi}_t^j + \lambda_{l,t}^j (1-\tau_t) U_c(c_t^j, l_t^j)/l_t^j \right) \frac{\partial}{\partial \tau_t} \left((y_t^j l_t^j)^{1-\tau_t} \right) (dj)$$
$$- \int_j \lambda_{l,t}^j (y_t^j l_t^j)^{1-\tau_t} (U_c(c_t^j, l_t^j)/l_t^j) \ell(dj).$$

Using $\frac{\partial}{\partial \tau_t}\left((y_t^j l_t^j)^{1-\tau_t}\right) = -\ln(y_t^j l_t^j)(y_t^j l_t^j)^{1-\tau_t}$, we finally deduce:

$$
\begin{aligned}
0 = & \int_j (y_t^j l_t^j)^{1-\tau_t}\left(\hat{\psi}_t^j + \lambda_{l,t}^j(1-\tau_t)U_c(c_t^j, l_t^j)/l_t^j\right)\ln(y_t^j l_t^j)(dj) \\
& + \int_j \lambda_{l,t}^j (y_t^j l_t^j)^{1-\tau_t}(U_c(c_t^j, l_t^j)/l_t^j)\ell(dj).
\end{aligned}
$$

Summary of FOCs.

$$
\begin{aligned}
\hat{\psi}_t^i &= \beta \mathbb{E}_t\left[(1+r_{t+1})\hat{\psi}_{t+1}^i\right], \\
\psi_{l,t}^i &= (1-\tau_t)w_t y_t^i (y_t^i l_t^i)^{-\tau_t}\hat{\psi}_t^i \\
&\quad + \mu_t F_{L,t} y_t^i - \lambda_{l,t}^i (1-\tau_t)\tau_t w_t y_t^i (y_t^i l_t^i)^{-\tau_t} U_c(c_t^i, l_t^i)/l_t^i, \\
\mu_t &= \beta(1+\tilde{r}_{t+1})\mu_{t+1}, \\
0 &= \int_j \left(\hat{\psi}_t^j a_{t-1}^j + \lambda_{c,t-1}^j U_c(c_t^j, l_t^j)\right)\ell(dj), \\
0 &= \int_j (y_t^j l_t^j)^{1-\tau_t}\left(\hat{\psi}_t^j + \lambda_{l,t}^j (1-\tau_t)U_c(c_t^j, l_t^j)/l_t^j\right)\ell(dj), \\
0 &= \int_j (y_t^j l_t^j)^{1-\tau_t}\left(\hat{\psi}_t^j + \lambda_{l,t}^j (1-\tau_t)U_c(c_t^j, l_t^j)/l_t^j\right)\ln(y_t^j l_t^j)(dj) \\
&\quad + \int_j \lambda_{l,t}^j (y_t^j l_t^j)^{1-\tau_t}(U_c(c_t^j, l_t^j)/l_t^j)\ell(dj).
\end{aligned}
$$

B.1 Checking that FOCs are identical

We check here that the first-order conditions of the Ramsey program derived in the general case of Section 2.6 exactly simplify to the first-order conditions derived in the specific case of Section 3. We start with expressing ψ_t^i and $\psi_{l,t}^i$ (equations (57) and (61)) in the context of the GHH utility function. We denote $C = c - \chi^{-1}\frac{l^{1+1/\varphi}}{1+1/\varphi}$. Since $U(c,l) = \ln\left(c - \chi^{-1}\frac{l^{1+1/\varphi}}{1+1/\varphi}\right)$, we compute:

$$
\begin{aligned}
U_c(c,l) &= \frac{1}{C},\ U_{cc}(c,l) = -\frac{1}{C^2},\ U_l(c,l) = -\chi^{-1}l^{1/\varphi}\frac{1}{C}, \\
U_{ll}(c,l) &= -\frac{\chi^{-1}l^{1/\varphi-1}}{C}\left(\frac{1}{\varphi} + \frac{\chi^{-1}l^{1/\varphi}}{C}\right),\ U_{cl}(c,l) = \frac{\chi^{-1}l^{1/\varphi}}{C^2}.
\end{aligned}
$$

Plugging this into equations (57) and (61) and using the labor Euler equation (11) stating that $\chi^{-1}l_t^{i,1/\varphi} = y_t^i w_t$, we deduce that the expressions of ψ_t^i and $\psi_{l,t}^i$ become:

$$
\psi_t^i C_t^i = 1 + \left(\lambda_{c,t}^i - R_t\lambda_{c,t-1}^i\right)\frac{1}{C_t^i}, \tag{171}
$$

$$
\psi_{l,t}^i C_t^i = y_t^i w_t\left(1 + \frac{\lambda_{l,t}^i}{\varphi l_t^i} + (\lambda_{c,t}^i - R_t\lambda_{c,t-1}^i)\frac{1}{C_t^i}\right). \tag{172}
$$

We now turn to the FOCs. Note that FOC (62) is exactly the same as FOC (32), while FOC (65) has no equivalent in the simplified version since the progressivity parameter τ_t is set to zero. FOC (60) can also be written with $\tau_t = 0$: $\psi^i_{l,t} = w_t y^i_t \psi^i_t + \mu_t (F_{L,t} - w_t) y^i_t$. Plugging (171) and (172) yields:

$$\frac{\lambda^i_{l,t}}{\varphi l^i_t} \frac{y^i_t w_t}{C^i_t} = \mu_t (F_{L,t} - w_t) y^i_t,$$
$$\frac{\lambda^i_{l,t}}{\varphi l^i_t} \frac{y^i_t}{C^i_t} = \mu_t (\frac{F_{L,t}}{w_t} - 1) y^i_t,$$

which is equivalent to $0 = 0$ for unemployed agents since their productivity is null. For employed agents with a productivity normalized to one, it becomes:

$$\lambda_{e,l,t} = \varphi \mu_t l_{e,t} C_{e,t} \frac{\tau^L_t}{1 - \tau^L_t}. \tag{173}$$

The three remaining FOCs are equations (59), (63), and (64). Taking advantage of the deterministic transitions between employment and unemployment, as well as the fact that unemployed agents are credit-constrained (implying $a_{u,t-1} = \lambda_{u,c,t-1} = 0$) with null productivity, these three FOCs can also be written as follows ($a_{e,t-1}, l_{e,t} > 0$):

$$\psi_{e,t} - \mu_t = \beta R_{t+1} (\psi_{u,t+1} - \mu_{t+1}), \tag{174}$$
$$\mu_t C_{u,t} = \psi_{u,t} C_{u,t} + \frac{\lambda_{e,c,t-1}}{a_{e,t-1}}, \tag{175}$$
$$\mu_t C_{e,t} = \psi_{e,t} C_{e,t} + \frac{\lambda_{e,l,t}}{l_{e,t}}, \tag{176}$$

while similarly expressions of ψ^i_t in (171) can further be specified as:

$$\psi_{e,t} C_{e,t} = 1 + \frac{\lambda_{e,c,t}}{C_{e,t}}, \tag{177}$$
$$\psi_{u,t} C_{u,t} = 1 - R_t \lambda_{e,c,t-1} \frac{1}{C_{u,t}}. \tag{178}$$

Combining (175) and (178) with $a_{e,t-1} = \frac{C_{u,t}}{R_t}$ (which is unemployed agents' budget constraint (17)) implies:

$$\mu_t C_{u,t} = 1, \tag{179}$$

with the expression of $C_{u,t} = R_t \frac{\beta}{1+\beta} \frac{w_{t-1} (\chi w_{t-1})^\varphi}{1+\varphi}$ identical to FOC (33).

Using the consumption Euler equation (21) stating that $\frac{1}{C_{e,t}} = \beta R_{t+1} \frac{1}{C_{u,t+1}}$, the budget constraints (16) and (17) implying that $C_{u,t} = \beta R_t C_{e,t-1}$, and (179) meaning that $1 = \beta \mu_{t+1} R_{t+1} C_{e,t}$, we deduce from (174) and (177):

$$\frac{\lambda_{e,c,t}}{C_{e,t}} = \frac{\beta}{1+\beta}(\mu_t C_{e,t} - 1). \tag{180}$$

Finally, we turn to FOC (176). Combined with the expressions of $\lambda_{e,l,t}$ in (173), $\psi_{e,t}$ in (177), and of $\lambda_{e,c,t}$ in (180), this becomes:

$$C_{e,t}\mu_t \left(1 - (1+\beta)\varphi \frac{\tau_t^L}{1-\tau_t^L}\right) = 1. \tag{181}$$

Using the budget constraint (16) stating that $C_{e,t} = \frac{w_t(\chi w_t)^{\varphi}}{(1+\beta)(1+\varphi)}$, equation (181) becomes FOC (31). This completes the proof that the generic FOCs of Section 2.6 exactly imply the FOCs (31)– (33).

C The Ramsey program on the truncated model

C.1 Formulation

We define the set of $(\xi_{y^N}^{u,0})_{y^N}$ such that:

$$\sum_{y^t \in \mathcal{Y}^t \left| (y_{t-N+1}^t, \ldots, y_t^t) = y^N \right.} u(c_t(y^t)) = \xi_{y^N}^{u,0} u\Bigg(\sum_{y^t \in \mathcal{Y}^t \left| (y_{t-N+1}^t, \ldots, y_t^t) = y^N \right.} c_t(y^t) \Bigg),$$

or compactly:

$$\xi_{y^N}^{u,0} u(c_{t,y^N}) := \sum_{y^N} u(c_t^i).$$

Similarly, we define $(\xi_{y^N}^{v,0})$, $(\xi_{y^N}^{u,1})$, $(\xi_{y^N}^{\tau})$, and $(\xi_{y^N}^{v,1})$ such that:

$$\begin{aligned}
\xi_{y^N}^{v,0} v(l_{t,y^N}) &:= \sum_{y^N} v(l_t^i), \\
\xi_{y^N}^{u,1} u'(c_{t,y^N}) &:= \sum_{y^N} u'(c_t^i), \\
\xi_{y^N}^{\tau} \sum_{y^N} (l_{t,y^N})^{\tau_t} &:= \sum_{y^N} (l_t^i)^{\tau_t}, \\
\xi_{t,s^N}^{v,1} v'(l_{t,y^N}) &:= \tau_t w_t \xi_{y^N}^{\tau} (l_{t,y^N} y_{y^N})^{1-\tau_t} \xi_{y^N}^{u,1} (u'(c_{t,y^N})/l_{t,y^N}).
\end{aligned}$$

The Ramsey problem can then be written as:

$$\max_{\left(r_t,\tilde{w}_t,\tilde{r}_t,\tau_t^K,\tau_t,\kappa_t,B_t,K_t,L_t,(a_t^i,c_t^i,l_t^i,\nu_t^i)_i\right)_{t\geq 0}} \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\sum_{y^N}S_{t,y^N}\omega_{y^N}(\xi_{y^N}^{u,0}u(c_{t,y^N})-\xi_{y^N}^{v,0}v(l_{t,y^N}))\right],$$

$$G_t+T_t+(1+r_t)B_{t-1}+r_tK_{t-1}+w_t\xi_{y^N}^{y}\sum_{y^N}(l_{t,y^N}y_{y^N})^{1-\tau_t}\ell(di)=F(K_{t-1},L_t,z_t)+B_t,$$

$$\begin{aligned}
&\text{for all } y^N\in\mathcal{Y}\text{: } ,c_{t,y^N}+a_{t,y^N}=w_t\xi_{y^N}^{y}(l_{t,y^N}y_{y^N})^{1-\tau_t}+(1+r_t)\tilde{a}_{t,y^N}+T_t,\\
&\quad a_{t,y^N}\geq-\bar{a},\ \nu_{t,y^N}(a_{t,y^N}+\bar{a})=0,\ \nu_{t,y^N}\geq 0,\\
&\quad \xi_{y^N}^{u,E}u'(c_{t,y^N})=\beta\mathbb{E}_t\Big[\sum_{\tilde{y}^N\in\mathcal{Y}^N}\Pi_{t,y^N\tilde{y}^N}\xi_{\tilde{y}^N}^{u,E}u'(c_{t+1,\tilde{y}^N})(1+r_{t+1})\Big]+\nu_{t,y^N},\\
&\quad \xi_{y^N}^{v,1}v'(l_{t,y^N})\equiv\tau_tw_t\xi_{y^N}^{y}(l_{t,y^N}y_{y^N})^{1-\tau_t}\xi_{y^N}^{u,1}(u'(c_{t,y^N})/l_{t,y^N}),\\
&\quad K_t+B_t=\sum_{y^N}S_{t,y^N}a_{t,y^N},\ L_t=\sum_{y^N}S_{t,y^N}y_{y^N}l_{t,y^N}.
\end{aligned}$$

C.2 Factorization

We now factorize the Ramsey problem of Section C.1. We define:

$$\begin{aligned}
J=&\mathbb{E}_0\sum_{t=0}^{\infty}\beta^t\sum_{y^N\in\mathcal{Y}}\Big[S_{t,y^N}\Big(\Big(\omega_{y^N}\xi_{y^N}^{u,0}u(c_{t,y^N})-\xi_{y^N}^{v,0}v(l_{t,y^N})\Big)\\
&-\Big(\lambda_{c,t,y^N}-\tilde{\lambda}_{c,t,y^N}(1+r_t)\Big)\xi_{y^N}^{u,1}U_c(c_{t,y^N},l_{t,y^N})\Big),\\
&-\lambda_{l,t,y^N}\Big(v'(l_{t,y^N})-\tau_tw_t(y_{t,y^N})^{\tau_t}\xi_{y^N}^{y}(l_{t,y^N})^{-\tau_t}\xi_{y^N}^{u,1}u'(c_{t,y^N})\Big)\Big].
\end{aligned}$$

The Ramsey program becomes maximizing J subject to the following constraints:

$$G_t+T_t+(1+r_t)B_{t-1}+r_tK_{t-1}+w_t\sum_{y^N}S_{y^N}\xi_{y^N}^{\tau}(l_{t,y^N}y_{y^N})^{1-\tau_t}=F(K_{t-1},L_t)+B_t$$

$$\begin{aligned}
&\text{for all } y^N\in\mathcal{Y}\text{: } ,c_{t,y^N}+a_{t,y^N}=w_t\xi_{y^N}^{y}(l_{t,y^N}y_{y^N})^{1-\tau_t}+(1+r_t)\tilde{a}_{t,y^N}+T_t,\\
&\quad a_{t,y^N}\geq-\bar{a},\ \nu_{t,y^N}(a_{t,y^N}+\bar{a})=0,\ \nu_{t,y^N}\geq 0,\\
&\quad K_t+B_t=\sum_{y^N}S_{t,y^N}a_{t,y^N},\ L_t=\sum_{y^N}S_{t,y^N}y_{t,y^N}^{i}l_{t,y^N}.
\end{aligned}$$

C.3 FOCs of the planner

Before expressing the FOCs of the Ramsey program, we define:

$$\begin{aligned}
\hat{\psi}_{t,y^N}:=&\ \omega_{y^N}\xi_{y^N}^{u,0}u'(c_{t,y^N})-\mu_t\\
&-\Big(\lambda_{c,t,y^N}\xi_{y^N}^{u,E}-(1+r_t)\tilde{\lambda}_{c,t,y^N}\xi_{y^N}^{u,E}-\lambda_{l,t,y^N}\xi_{y^N}^{y}\tau_tw_t(y_0^N)^{1-\tau_t}l_{t,y^N}^{-\tau_t}\xi_{y^N}^{u,1}\Big)u''(c_{t,y^N}).
\end{aligned}$$

The two Euler equations can be written as follows:

$$\xi^{u,E}_{y^N} u'(c_{t,y^N}) = \beta \mathbb{E}_t \Big[\sum_{\tilde{y}^N \in \mathcal{Y}^N} \Pi_{t,y^N \tilde{y}^N} \xi^{u,E}_{\tilde{y}^N} u'(c_{t+1,\tilde{y}^N})(1 + r_{t+1}) \Big] + \nu_{t,y^N},$$

$$\xi^{v,1}_{t,s^N} v'(l_{t,y^N}) = (1 - \tau_t)\, w_t \xi^{\tau}_{y^N} (l_{t,y^N} y_{y^N})^{1-\tau_t} \xi^{u,1}_{y^N} (u'(c_{t,y^N})/l_{t,y^N}),$$

while the constraints of the Ramsey program become:

$$\begin{aligned}
B_t + K^{\alpha}_{t-1} L^{1-\alpha}_t &= G_t + T_t + (1-\delta) B_{t-1} + (r_t + \delta) A_{t-1} \\
&\quad + w_t \sum_{y^N \in \mathcal{Y}^N} S_{t,y^N} \xi^{\tau}_{y^N} (y_{y^N} l_{t,y^N})^{1-\tau_t}, \\
\tilde{\lambda}_{t,y^N} &= \frac{1}{S_{t,y^N}} \sum_{\tilde{y}^N \in \mathcal{Y}^N} S_{t-1,\tilde{y}^N} \lambda_{t-1,\tilde{y}^N} \Pi_{t,\tilde{y}^N,y^N}, \\
c_{t,y^N} + a_{t,y^N} &= w_t (l_{t,y^N} y_{y^N})^{1-\tau_t} + (1 + r_t) \tilde{a}_{t,y^N} + T_t, \\
a_{t,y^N} \geq 0 \text{ and } \tilde{a}_{t,y^N} &= \sum_{\tilde{y}^N \in \mathcal{Y}^N} \Pi_{\tilde{y}^N y^N,t} \frac{S_{t-1,\tilde{y}^N}}{S_{t,y^N}} a_{t-1,\tilde{y}^N}.
\end{aligned}$$

The FOCs of the Ramsey program can finally be written as follows:

$$\begin{aligned}
\hat{\psi}_{t,y^N} &= \beta \mathbb{E}_t \Big[(1 + r_{t+1}) \sum_{\tilde{y}^N \in \mathcal{Y}^N} \Pi_{t,y^N \tilde{y}^N} \hat{\psi}_{t+1,\tilde{y}^N} \Big] \text{ if } \nu_{y^N} = 0 \text{ and } \lambda_{t,y^N} = 0 \text{ otherwise}, \\
\hat{\psi}_{t,y^N} &= \frac{1}{\tau_t w_t \xi^{\tau}_{y^N} (y^N_0)^{1-\tau_t} l^{\tau_t}_{t,y^N}} (\omega_{y^N} \xi^{v,0}_{y^N} v'(l_{t,y^N}) + \lambda_{l,t,y^N} \xi^{v,1}_{y^N} v''(l_{t,y^N})) \\
&\quad + \lambda_{l,t,y^N} \tau_t \xi^{u,1}_{y^N} (u'(c_{t,y^N})/l_{t,y^N}) \\
&\quad - \mu_t (1-\alpha) \frac{Y_t}{\tau_t w_t \xi^{\tau}_{y^N} (y^N_0)^{-\tau_t} l^{-\tau_t}_{t,y^N} L_t}, \\
0 &= \sum_{y^N \in \mathcal{Y}} S_{y^N} \left(\hat{\psi}_{t,y^N} \tilde{a}_{t,y^N} + \tilde{\lambda}_{c,t,y^N} \xi^{u,E}_{y^N} u'(c_{t,y^N}) \right), \\
0 &= \sum_{y^N \in \mathcal{Y}} S_{y^N} \xi^{\tau}_{y^N} (l_{t,y^N} y_{y^N})^{\tau_t} \left(\hat{\psi}_{t,y^N} + \lambda_{l,t,y^N} (1 - \tau_t) \xi^{u,1}_{y^N} (u'(c_{t,y^N})/l_{t,y^N}) \right), \\
\mu_t &= \beta \mathbb{E} \left[\mu_{t+1} \left(1 + \alpha \frac{Y_{t+1}}{K_t} - \delta \right) \right], \\
0 &= \sum_{y^N \in \mathcal{Y}} S_{y^N} \lambda_{l,t,y^N} \xi^{\tau}_{y^N} (l_{t,y^N} y_{y^N})^{1-\tau_t} \xi^{u,1}_{y^N} (u'(c_{t,y^N})/l_{t,y^N}) \\
&\quad + \sum_{y^N \in \mathcal{Y}} S_{y^N} \left(\hat{\psi}_{t,y^N} + \lambda_{l,t,y^N} (1 - \tau_t) \xi^{u,1}_{y^N} (u'(c_{t,y^N})/l_{t,y^N}) \right) \ln \left(l_{t,y^N} y_{y^N} \right) \xi^{\tau}_{y^N} (l_{t,y^N} y_{y^N})^{1-\tau_t}.
\end{aligned}$$

D Matrix expression

In this section, we provide closed-form formulas for preference multipliers ξs (Section D.1) and the Pareto weights ωs. The process to estimate the Pareto weight for the steady-state to be optimal is provided in Section D.4.

We start with some notations:

$\circ$ is the Hadamard product, $\otimes$ is the Kronecker product, $\times$ is the usual matrix product.

For any vector V, we denote by $diag\,(V)$ the diagonal matrix with V on the diagonal.

The matrix representation consists in stacking together the equations characterizing the steady state, so as to provide a convenient matrix notation for solving the steady state. It also provides an efficient solution to compute the values for the coefficients (ξ_{y^N}) and (ω_{y^N}). The starting point is to observe that a history y^N can be seen as an N-length vector $\{y_{-N+1}, \ldots, y_0\}$ of elements of $\mathcal{Y}$. The number of histories is $N_{tot} = Y^N$. We can identify each history by an integer $k_{y^N} = 1, \ldots, N_{tot}$:

$$k_{y^N} = \sum_{k=0}^{N-1} N_{tot}^{-N+1-k}(y_k - 1) + 1, \tag{182}$$

which corresponds to an enumeration in base Y.

D.1 A closed-form formula for the ξs

Let $\mathbf{S}$ be the N_{tot}-vector of steady-state history sizes that is defined as $\boldsymbol{S} = (S_{k_{y^N}})_{k_{y^N}=1,\ldots,N_{tot}}$, by stacking history sizes for all histories using the enumeration given by (182). Similarly, let $\boldsymbol{a}$, $\boldsymbol{c}$, $\boldsymbol{l}$, and $\boldsymbol{\nu}$ be the N_{tot}-vectors of end-of-period wealth, consumption, labor supply, and Lagrange multipliers, respectively. These vectors are known from the steady-state equilibrium of the Bewley model. Each element is defined as the truncation of the relevant variable. We also define:

$$\mathbf{W} = w \begin{bmatrix} y_1 \\ \vdots \\ y_Y \end{bmatrix} \otimes 1_B,\ \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_Y \end{bmatrix} \otimes 1_B,$$

where 1_B is a vector of 1 of length Y^{N-1}. We define $\mathbb{P}$ as the diagonal matrix having 1 on the diagonal at y^N if and only if the history y^N is not credit-constrained (i.e., $\nu_{y^N} = 0$), and 0 otherwise. We similarly define $\boldsymbol{P}^c = \boldsymbol{I} - \boldsymbol{P}$, where $\boldsymbol{I}$ is the $(N_{tot} \times N_{tot})$-identity matrix. Noting $\boldsymbol{\Pi}$ as the transition matrix across histories, we obtain the following steady-state relationships.

D.2 Matrix expressions

The market allocation for the truncated model can be represented as:

$$\begin{aligned}
S_{y^N} &= \sum_{\hat{y}^N \in \mathcal{Y}^N} S_{\hat{y}^N} \Pi_{\hat{y}^N y^N}, \\
S_{y^N} \tilde{a}_{t,y^N} &= \sum_{\hat{y}^N \in \mathcal{Y}^N} S_{\hat{y}^N} \Pi_{\hat{y}^N y^N} a_{t-1,\hat{y}^N}, \\
c_{t,y^N} + a_{t,y^N} &= w_t \xi^y_{y^N} (l_{t,y^N} y_{y^N})^{\tau_t} + (1+r_t)\tilde{a}_{t,y^N} + T_t, \\
\xi^{u,E}_{y^N} u'(c_{t,y^N}) &= \beta \mathbb{E}_t \Big[\sum_{\tilde{y}^N \in \mathcal{Y}^N} \Pi_{t,y^N \tilde{y}^N} \xi^{u,E}_{\tilde{y}^N} u'(c_{t+1,\tilde{y}^N})(1+r_{t+1}) \Big] + \nu_{t,y^N}, \\
\xi^{v,1}_{t,s^N} v'(l_{t,y^N}) &= \tau_t w_t \xi^y_{y^N} (l_{t,y^N} y_{y^N})^{\tau_t} \xi^{u,1}_{y^N} (u'(c_{t,y^N})/l_{t,y^N}).
\end{aligned}$$

These equations have a direct matrix expression:

$$\boldsymbol{S} = \Pi \boldsymbol{S}, \tag{183}$$

$$\boldsymbol{S} \circ \boldsymbol{c} + \boldsymbol{S} \circ \boldsymbol{a} = (1+r)\Pi \left(\boldsymbol{S} \circ \boldsymbol{a}\right) + w\boldsymbol{S} \circ \xi^y \circ (\boldsymbol{y} \circ \boldsymbol{l})^\tau + T\mathbf{1}, \tag{184}$$

$$(\boldsymbol{I} - \boldsymbol{P})\boldsymbol{a} = \mathbf{0}_{N_{tot} \times 1}. \tag{185}$$

The Euler is:

$$\boldsymbol{\xi}^{u,E} \circ u'(\boldsymbol{c}) = \beta(1+r)\boldsymbol{\Pi}^\top \left(\boldsymbol{\xi}^{u,E} \circ u'(\boldsymbol{c}) \right) + \boldsymbol{\nu},$$

where the matrix $\Pi^\top$ (the transpose of Π) is used to make expectations about next-period histories.

Computing aggregation parameters. The previous equation can be written as:

$$\boldsymbol{D}_{u'(\boldsymbol{c})} \boldsymbol{\xi}^{u,E} = \beta(1+r)\boldsymbol{\Pi}^\top \boldsymbol{D}_{u'(\boldsymbol{c})} \boldsymbol{\xi}^{u,E} + \boldsymbol{\nu},$$

Finally, the vector of parameters such that the Euler equations are fulfilled are:

$$\boldsymbol{\xi}^{u,E} = \left[\left(\boldsymbol{I} - \beta(1+r)\boldsymbol{\Pi}^\top \right) \boldsymbol{D}_{u'(\boldsymbol{c})} \right]^{-1} \boldsymbol{\nu}, \tag{186}$$

and the labor supply equation $\xi^{v,1}_{t,s^N} v'(l_{t,y^N}) = \tau_t w_t (l_{t,y^N} y_{y^N})^{\tau_t} \xi^y_{y^N} \xi^{u,1}_{y^N} (u'(c_{t,y^N})/l_{t,y^N})$ gives

$$\boldsymbol{\xi}^{v,1} = \tau w (\boldsymbol{y} \circ \boldsymbol{l})^\tau \circ \boldsymbol{\xi}^y \circ \boldsymbol{\xi}^{u,1} \circ u'(\boldsymbol{c})./(\boldsymbol{l} \circ v'(\boldsymbol{l})),$$

and

$$\begin{aligned}
\tilde{\boldsymbol{\xi}}^{v,1} &= \boldsymbol{\xi}^{v,1}./(\tau w \boldsymbol{\xi}^y \circ \boldsymbol{y}^\tau \circ \boldsymbol{l}^{\tau-1}) \\
\tilde{\boldsymbol{\xi}}^{v,0} &= \boldsymbol{\xi}^{v,0}./(\tau w \boldsymbol{\xi}^y \circ \boldsymbol{y}^\tau \circ \boldsymbol{l}^{\tau-1}).
\end{aligned}$$

To simplify the algebra below, define:

$$\tilde{\boldsymbol{\xi}}^{u,1} = \boldsymbol{\xi}^{u,1}./\boldsymbol{l}.$$

D.3 Finding the constraint on the Pareto weights ω

We now construct the constraints that the Pareto weights ω must fulfill for the steady-state allocation to be optimal for the observed instruments of the planner. Formally, all the first-order conditions of the planer must be fulfilled. We show that there are two vectors $\boldsymbol{L}_2, \boldsymbol{L}_3$ such that all first-order conditions of the planner are fulfilled when $\boldsymbol{L}_2\bar{\boldsymbol{\omega}} = 0$ and $\boldsymbol{L}_3\bar{\boldsymbol{\omega}} = 0$. The derivation of these vectors is not complicated, but tedious. We provide the detailed calculations and use these vectors in the estimation procedure in Section D.4.

First, from the dynamics of Lagrange multipliers $S_{t,y^N}\tilde{\lambda}_{c,t,y^N} = \sum_{\tilde{y}^N \in \mathcal{Y}^N} S_{t-1,\tilde{y}^N}\lambda_{c,t-1,\tilde{y}^N}\Pi_{t,\tilde{y}^N,y^N}$, we have:

$$\boldsymbol{S} \circ \tilde{\boldsymbol{\lambda}}_c = \boldsymbol{\Pi}(\boldsymbol{S} \circ \boldsymbol{\lambda}_c).$$

The first-order conditions of the planner at the steady state can respectively be written as follows using matrix notation:

$$\begin{aligned}
\hat{\psi}_{t,y^N} &= \omega_{y^N}\xi^{u,0}_{y^N}u'(c_{t,y^N}) \\
&\quad - \left(\lambda_{c,t,y^N}\xi^{u,E}_{y^N} - (1+r_t)\tilde{\lambda}_{c,t,y^N}\xi^{u,E}_{y^N} - \lambda_{l,t,y^N}\tau_t w_t \xi^y_{y^N}(y_0^N)^{\tau_t} l^{\tau_t-1}_{t,y^N}\xi^{u,1}_{y^N}\right) u''(c_{t,y^N}) - \mu_t. \\
0 &= \hat{\psi}_{t,y^N} + \lambda_{l,t,y^N}(\tau_t - 1)\xi^{u,1}_{y^N}(u'(c_{t,y^N})/l_{t,y^N}) + \mu_t(1-\alpha)\frac{Y_t}{\tau_t w_t \xi^y_{y^N}(y_0^N)^{\tau_t-1} l^{\tau_t-1}_{t,y^N} L_t} \\
&\quad - \frac{1}{\tau_t w_t \xi^y_{y^N}(y_0^N)^{\tau_t} l^{\tau_t-1}_{t,y^N}}(\omega_{y^N}\xi^{v,0}_{y^N}v'(l_{t,y^N}) + \lambda_{l,t,y^N}\xi^{v,1}_{y^N}v''(l_{t,y^N})) \\
0 &= \sum_{y^N \in \mathcal{Y}} S_{y^N}\left(\hat{\psi}_{t,y^N}\tilde{a}_{t,y^N} + \tilde{\lambda}_{c,t,y^N}\xi^{u,E}_{y^N}u'(c_{t,y^N})\right) \\
0 &= \sum_{y^N \in \mathcal{Y}} S_{y^N}\xi^y_{y^N}(l_{t,y^N}y_{y^N})^{\tau_t}\left(\hat{\psi}_{t,y^N} + \lambda_{l,t,y^N}\tau_t\xi^{u,1}_{y^N}(u'(c_{t,y^N})/l_{t,y^N})\right) \\
\mu_t &= \beta\mathbb{E}\left[\mu_{t+1}\left(1 + \alpha\frac{Y_{t+1}}{K_t} - \delta\right)\right] \\
0 &= \sum_{y^N \in \mathcal{Y}} S_{y^N}\left(\hat{\psi}_{t,y^N} + \lambda_{l,t,y^N}\tau_t\xi^{u,1}_{y^N}(u'(c_{t,y^N})/l_{t,y^N})\right)\ln\left(l_{t,y^N}y_{y^N}\right)\xi^y_{y^N}(l_{t,y^N}y_{y^N})^{\tau_t} \\
&\quad + \sum_{y^N \in \mathcal{Y}} S_{y^N}\lambda_{l,t,y^N}\xi^y_{y^N}(l_{t,y^N}y_{y^N})^{\tau_t}\xi^{u,1}_{y^N}(u'(c_{t,y^N})/l_{t,y^N})
\end{aligned}$$

We find the corresponding matrix representation:

$$
\begin{aligned}
\hat{\boldsymbol{\psi}} &= \boldsymbol{\omega} \circ \boldsymbol{\xi}^{u,0} \circ u'(\boldsymbol{c}) \\
&\quad - \Big(\boldsymbol{\lambda}_c \circ \boldsymbol{\xi}^{u,E} - (1+r)\tilde{\boldsymbol{\lambda}}_c \circ \boldsymbol{\xi}^{u,E} - \tau w \boldsymbol{\lambda}_l \circ \boldsymbol{\xi}^{y} \circ (\boldsymbol{y} \circ \boldsymbol{l})^{\tau} \circ (\boldsymbol{\xi}^{u,1}./\boldsymbol{l})\Big) \circ u''(\boldsymbol{c}) - \mu \mathbf{1} \\
\boldsymbol{P}\hat{\boldsymbol{\psi}} &= \beta(1+r)\boldsymbol{P}\Pi^{\top}\hat{\boldsymbol{\psi}}, \\
(\boldsymbol{I} - \boldsymbol{P})\boldsymbol{\lambda}_c &= 0, \\
0 &= \boldsymbol{S} \circ \hat{\boldsymbol{\psi}} + (\tau - 1)\boldsymbol{\xi}^{u,1} \circ (\boldsymbol{S} \circ \boldsymbol{\lambda}_l) \circ u'(\boldsymbol{c})./\boldsymbol{l} \\
&\quad + \mu F_L \boldsymbol{S}./(\tau w \boldsymbol{\xi}^{y} \circ \boldsymbol{y}^{\tau-1} \circ \boldsymbol{l}^{\tau-1}) \\
&\quad - \boldsymbol{S} \circ \boldsymbol{\omega} \circ \tilde{\boldsymbol{\xi}}^{v,0} \circ v'(\boldsymbol{l}) + \boldsymbol{S} \circ \boldsymbol{\lambda}^{l} \circ \tilde{\boldsymbol{\xi}}^{v,1} \circ v''(\boldsymbol{l}) \\
\tilde{\boldsymbol{a}}^{\top} \times (\boldsymbol{S} \circ \hat{\boldsymbol{\psi}}) &= -\left(\boldsymbol{\xi}^{u,E} \circ u'(\boldsymbol{c})\right)^{\top} \times \left(\boldsymbol{S} \circ \tilde{\boldsymbol{\lambda}}_c\right), \\
(\boldsymbol{\xi}^{y} \circ (\boldsymbol{y} \circ \boldsymbol{l})^{\tau})^{\top} \times (\boldsymbol{S} \circ \hat{\boldsymbol{\psi}}) &= -\tau \left(\boldsymbol{\xi}^{y} \circ (\boldsymbol{y} \circ \boldsymbol{l})^{\tau} \circ \boldsymbol{\xi}^{u,1} \circ (u'(\boldsymbol{c})./\boldsymbol{l})\right)^{\top} \times (\boldsymbol{S} \circ \boldsymbol{\lambda}_l), \\
1 &= \beta(F_K + 1), \\
0 &= (\ln(\boldsymbol{y} \circ \boldsymbol{l}) \circ \boldsymbol{\xi}^{y} \circ (\boldsymbol{y} \circ \boldsymbol{l})^{\tau})^{\top} \times (\boldsymbol{S} \circ \hat{\boldsymbol{\psi}}) \\
&\quad + \Big((\mathbf{1} + \tau \ln(\boldsymbol{y} \circ \boldsymbol{l})) \circ \boldsymbol{\xi}^{y} \circ (\boldsymbol{y} \circ \boldsymbol{l})^{\tau} \circ \boldsymbol{\xi}^{u,1} \circ u'(\boldsymbol{c})./\boldsymbol{l}\Big)^{\top} \times (\boldsymbol{S} \circ \boldsymbol{\lambda}_l)
\end{aligned}
$$

To simplify the expressions, define:

$$
\begin{aligned}
\bar{\lambda}^{l} &:= \mathbf{S} \circ \lambda^{l} \\
\bar{\psi} &:= \mathbf{S} \circ \hat{\psi} \\
\bar{\Pi} &:= \mathbf{S} \circ \Pi^{\top} \circ (1./\mathbf{S}) \\
\bar{\omega} &:= \mathbf{S} \circ \omega \\
\bar{\lambda}_c &:= \mathbf{S} \circ \lambda_c \\
\mathbf{S} \circ \tilde{\lambda}_c &:= \Pi \bar{\boldsymbol{\lambda}}_c \\
\tilde{\boldsymbol{\xi}}^{u,1} &:= \boldsymbol{\xi}^{u,1}./\boldsymbol{l}
\end{aligned}
$$

These definitions give:

$$\bar{\boldsymbol{\psi}} = \bar{\boldsymbol{\omega}} \circ \boldsymbol{\xi}^{u,0} \circ u'(\boldsymbol{c}) \tag{187}$$
$$- \left(\bar{\boldsymbol{\lambda}}_c \circ \boldsymbol{\xi}^{u,E} - (1+r)\boldsymbol{\Pi}\bar{\boldsymbol{\lambda}}_c \circ \boldsymbol{\xi}^{u,E} - \tau w \bar{\boldsymbol{\lambda}}_l \circ \boldsymbol{\xi}^y \circ (\boldsymbol{y} \circ \boldsymbol{l})^\tau \circ \tilde{\boldsymbol{\xi}}^{u,1}\right) \circ u''(\boldsymbol{c}) \tag{188}$$
$$- \mu \boldsymbol{S} \tag{189}$$
$$\boldsymbol{P}\bar{\boldsymbol{\psi}} = \beta(1+r)\boldsymbol{P}\bar{\boldsymbol{\Pi}}\bar{\boldsymbol{\psi}}, \tag{190}$$
$$(\boldsymbol{I} - \boldsymbol{P})\bar{\boldsymbol{\lambda}}_c = 0, \tag{191}$$
$$\bar{\boldsymbol{\omega}} \circ \tilde{\boldsymbol{\xi}}^{v,0} \circ v'(\boldsymbol{l}) + \bar{\boldsymbol{\lambda}}_l \circ \tilde{\boldsymbol{\xi}}^{v,1} \circ v''(\boldsymbol{l}) = \bar{\boldsymbol{\psi}} + (\tau - 1)\tilde{\boldsymbol{\xi}}^{u,1} \circ u'(\boldsymbol{c}) \circ \bar{\boldsymbol{\lambda}}_l + \mu F_L \boldsymbol{S}./(\tau w \boldsymbol{\xi}^y \circ \boldsymbol{y}^{\tau-1} \circ \boldsymbol{l}^{\tau-1}) \tag{192}$$
$$\tilde{\boldsymbol{a}}^\top \bar{\boldsymbol{\psi}} = -\left(\boldsymbol{\xi}^{u,E} \circ u'(\boldsymbol{c})\right)^\top \Pi \bar{\boldsymbol{\lambda}}_c, \tag{193}$$
$$\left(\boldsymbol{\xi}^y \circ (\boldsymbol{y} \circ \boldsymbol{l})^\tau\right)^\top \bar{\boldsymbol{\psi}} = -\tau \left(\boldsymbol{\xi}^y \circ (\boldsymbol{y} \circ \boldsymbol{l})^\tau \circ \tilde{\boldsymbol{\xi}}^{u,1} \circ u'(\boldsymbol{c})\right)^\top \bar{\boldsymbol{\lambda}}_l, \tag{194}$$
$$\left(\ln(\boldsymbol{y} \circ \boldsymbol{l}) \circ \boldsymbol{\xi}^y \circ (\boldsymbol{y} \circ \boldsymbol{l})^\tau\right)^\top \bar{\boldsymbol{\psi}} = -\left((\boldsymbol{1} + \tau \ln(\boldsymbol{y} \circ \boldsymbol{l})) \circ \boldsymbol{\xi}^y \circ (\boldsymbol{y} \circ \boldsymbol{l})^\tau \circ \tilde{\boldsymbol{\xi}}^{u,1} \circ u'(\boldsymbol{c})\right)^\top \bar{\boldsymbol{\lambda}}_l \tag{195}$$

This is a system in $\bar{\psi}, \bar{\omega}, \bar{\lambda}, \bar{\theta}^l, \mu$.

D.3.1 Solving for $\bar{\lambda}$ and not $\bar{\psi}$

$$\bar{\boldsymbol{\omega}} \circ \tilde{\boldsymbol{\xi}}^{v,0} \circ v'(\boldsymbol{l}) + \bar{\boldsymbol{\lambda}}_l \circ \tilde{\boldsymbol{\xi}}^{v,1} \circ v''(\boldsymbol{l}) = \bar{\boldsymbol{\psi}} + (\tau - 1)\tilde{\boldsymbol{\xi}}^{u,1} \circ u'(\boldsymbol{c}) \circ \bar{\boldsymbol{\lambda}}_l + \mu F_L \boldsymbol{S}./(\tau w \boldsymbol{\xi}^y \circ \boldsymbol{y}^{\tau-1} \circ \boldsymbol{l}^{\tau-1})$$

Equation (192) yields:

$$\boldsymbol{D}_{\tilde{\boldsymbol{\xi}}^{v,1} \circ v''(\boldsymbol{l}) - (\tau-1)\tilde{\boldsymbol{\xi}}^{u,1} \circ u'(\boldsymbol{c})} \bar{\boldsymbol{\lambda}}_l = \mu F_L \boldsymbol{S}./(\tau w \boldsymbol{\xi}^y \circ \boldsymbol{y}^\tau \circ \boldsymbol{l}^{\tau-1}) + \bar{\boldsymbol{\psi}} - \boldsymbol{D}_{\tilde{\boldsymbol{\xi}}^{v,0} \circ v'(\boldsymbol{l})} \bar{\boldsymbol{\omega}}$$
$$\bar{\boldsymbol{\lambda}}_l = \boldsymbol{M}_0 \bar{\boldsymbol{\omega}} + \boldsymbol{M}_1 \bar{\boldsymbol{\psi}} + \mu \boldsymbol{V}_0. \tag{196}$$

with:

$$\boldsymbol{M}_0 = -\boldsymbol{M}_1 \boldsymbol{D}_{\tilde{\boldsymbol{\xi}}^{v,0} \circ v'(\boldsymbol{l})},$$
$$\boldsymbol{M}_1 = \boldsymbol{D}^{-1}_{\tilde{\boldsymbol{\xi}}^{v,1} \circ v''(\boldsymbol{l}) - (\tau-1)\tilde{\boldsymbol{\xi}}^{u,1} \circ u'(\boldsymbol{c})},$$
$$\boldsymbol{V}_0 = F_L \boldsymbol{M}_1 \boldsymbol{S}./(\tau w \boldsymbol{\xi}^y \circ \boldsymbol{y}^{\tau-1} \circ \boldsymbol{l}^{\tau-1})$$

Then equation (187) implies:

$$\bar{\boldsymbol{\psi}} = \bar{\boldsymbol{\omega}} \circ \boldsymbol{\xi}^{u,0} \circ u'(\boldsymbol{c}) - \left(\bar{\boldsymbol{\lambda}}_c \circ \boldsymbol{\xi}^{u,E} - (1+r)\boldsymbol{\Pi}\bar{\boldsymbol{\lambda}}_c \circ \boldsymbol{\xi}^{u,E} - \tau w \bar{\boldsymbol{\lambda}}_l \circ \boldsymbol{\xi}^y \circ (\boldsymbol{y} \circ \boldsymbol{l})^\tau \circ \tilde{\boldsymbol{\xi}}^{u,1}\right) \circ u''(\boldsymbol{c}) - \mu \boldsymbol{S}$$
$$\bar{\boldsymbol{\psi}} = \boldsymbol{D}_{\boldsymbol{\xi}^{u,0} \circ u'(\boldsymbol{c})} \bar{\boldsymbol{\omega}} - \boldsymbol{D}_{\boldsymbol{\xi}^{u,E} \circ u''(\boldsymbol{c})} (\boldsymbol{I} - (1+r)\boldsymbol{\Pi}) \bar{\boldsymbol{\lambda}}_c + \tau w \boldsymbol{D}_{\boldsymbol{\xi}^y \circ (\boldsymbol{y} \circ \boldsymbol{l})^\tau \circ \tilde{\boldsymbol{\xi}}^{u,1} \circ u''(\boldsymbol{c})} \bar{\boldsymbol{\lambda}}_l - \mu \boldsymbol{S}$$
$$= \hat{\boldsymbol{M}}_0 \bar{\boldsymbol{\omega}} + \hat{\boldsymbol{M}}_1 \bar{\boldsymbol{\lambda}}_c + \hat{\boldsymbol{M}}_2 \bar{\boldsymbol{\lambda}}_l - \mu \boldsymbol{S} \tag{197}$$

with:

$$\hat{\boldsymbol{M}}_0 = \boldsymbol{D}_{\boldsymbol{\xi}^{u,0}\circ u'(\boldsymbol{c})},$$
$$\hat{\boldsymbol{M}}_1 = -\boldsymbol{D}_{\boldsymbol{\xi}^{u,E}\circ u''(\boldsymbol{c})}(\boldsymbol{I} - (1+r)\boldsymbol{\Pi}),$$
$$\hat{\boldsymbol{M}}_2 = \tau w \boldsymbol{D}_{\boldsymbol{\xi}^{y}\circ(\boldsymbol{y}\circ\boldsymbol{l})^{\tau}\circ\tilde{\boldsymbol{\xi}}^{u,1}\circ u''(\boldsymbol{c})}.$$

So using (197):

$$\bar{\boldsymbol{\psi}} = \hat{\boldsymbol{M}}_0\bar{\boldsymbol{\omega}} + \hat{\boldsymbol{M}}_1\bar{\boldsymbol{\lambda}}_c + \hat{\boldsymbol{M}}_2(\boldsymbol{M}_0\bar{\boldsymbol{\omega}} + \boldsymbol{M}_1\bar{\boldsymbol{\psi}} + \mu\boldsymbol{V}_0) - \mu\boldsymbol{S}$$
$$\left(\boldsymbol{I} - \hat{\boldsymbol{M}}_2\boldsymbol{M}_1\right)\bar{\boldsymbol{\psi}} = (\hat{\boldsymbol{M}}_0 + \hat{\boldsymbol{M}}_2\boldsymbol{M}_0)\bar{\boldsymbol{\omega}} + \hat{\boldsymbol{M}}_1\bar{\boldsymbol{\lambda}}_c + \mu(\hat{\boldsymbol{M}}_2\boldsymbol{V}_0 - \boldsymbol{S}).$$

We define:

$$\boldsymbol{M}_2 = \boldsymbol{I} - \hat{\boldsymbol{M}}_2\boldsymbol{M}_1$$
$$\boldsymbol{M}_3 = \boldsymbol{M}_2^{-1}(\hat{\boldsymbol{M}}_0 + \hat{\boldsymbol{M}}_2\boldsymbol{M}_0)$$
$$\boldsymbol{M}_4 = \boldsymbol{M}_2^{-1}\hat{\boldsymbol{M}}_1$$
$$\boldsymbol{V}_1 = \boldsymbol{M}_2^{-1}(\hat{\boldsymbol{M}}_2\boldsymbol{V}_0 - \boldsymbol{S})$$

Then:

$$\bar{\boldsymbol{\psi}} = \boldsymbol{M}_3\bar{\boldsymbol{\omega}} + \boldsymbol{M}_4\bar{\boldsymbol{\lambda}}_c + \mu\boldsymbol{V}_1. \tag{198}$$

Then, using (190), (191), and (198), we get:

$$(\boldsymbol{I} - \boldsymbol{P})\bar{\boldsymbol{\lambda}}_c + \boldsymbol{P}(I - \beta(1+r)\bar{\boldsymbol{\Pi}})\bar{\boldsymbol{\psi}} = 0$$
$$\left((\boldsymbol{I} - \boldsymbol{P}) + \boldsymbol{P}(I - \beta(1+r)\bar{\boldsymbol{\Pi}})\boldsymbol{M}_4\right)\bar{\boldsymbol{\lambda}}_c = -\boldsymbol{P}(I - \beta(1+r)\bar{\boldsymbol{\Pi}})\boldsymbol{M}_3\bar{\boldsymbol{\omega}} - \mu\boldsymbol{P}(I - \beta(1+r)\bar{\boldsymbol{\Pi}})\boldsymbol{V}_1$$

We define:

$$\tilde{\boldsymbol{R}}_5 = -((\boldsymbol{I} - \boldsymbol{P}) + \boldsymbol{P}(I - \beta(1+r)\bar{\boldsymbol{\Pi}})\boldsymbol{M}_4)^{-1}\boldsymbol{P}(I - \beta(1+r)\bar{\boldsymbol{\Pi}})$$
$$\boldsymbol{M}_5 = \tilde{\boldsymbol{R}}_5\boldsymbol{M}_3$$
$$\boldsymbol{V}_2 = \tilde{\boldsymbol{R}}_5\boldsymbol{V}_1$$

and get

$$\bar{\boldsymbol{\lambda}}_c = \boldsymbol{M}_5\bar{\boldsymbol{\omega}} + \mu\boldsymbol{V}_2. \tag{199}$$

Then we use equation (193):

$$\tilde{\boldsymbol{a}}^{\top}\bar{\boldsymbol{\psi}} + \left(\boldsymbol{\xi}^{u,E}\circ u'(\boldsymbol{c})\right)^{\top}\boldsymbol{\Pi}\bar{\boldsymbol{\lambda}}_c = 0$$

which becomes

$$\tilde{\boldsymbol{a}}^\top(\boldsymbol{M}_3\bar{\boldsymbol{\omega}} + \boldsymbol{M}_4\bar{\boldsymbol{\lambda}}_c + \mu\boldsymbol{V}_1) + \left(\boldsymbol{\xi}^{u,E} \circ u'(\boldsymbol{c})\right)^\top \Pi\bar{\boldsymbol{\lambda}}_c$$

$$\tilde{\boldsymbol{a}}^\top\left((\boldsymbol{M}_3 + \boldsymbol{M}_4\boldsymbol{M}_5)\bar{\boldsymbol{\omega}} + \mu(\boldsymbol{V}_1 + \boldsymbol{M}_4\boldsymbol{V}_2)\right) + (\boldsymbol{\xi}^{u,E} \circ u'(\boldsymbol{c}))^\top\boldsymbol{\Pi}(\boldsymbol{M}_5\bar{\boldsymbol{\omega}} + \mu\boldsymbol{V}_2) = 0$$

$$\left(\tilde{\boldsymbol{a}}^\top(\boldsymbol{M}_3 + \boldsymbol{M}_4\boldsymbol{M}_5) + (\boldsymbol{\xi}^{u,E} \circ u'(\boldsymbol{c}))^\top\boldsymbol{\Pi}\boldsymbol{M}_5\right)\bar{\boldsymbol{\omega}} + \mu(\tilde{\boldsymbol{a}}^\top\boldsymbol{V}_1 + \tilde{\boldsymbol{a}}^\top\boldsymbol{M}_4\boldsymbol{V}_2 + (\boldsymbol{\xi}^{u,E} \circ u'(\boldsymbol{c}))^\top\boldsymbol{\Pi}\boldsymbol{V}_2) = 0$$

We define:

$$C_1 = \tilde{\boldsymbol{a}}^\top(\boldsymbol{V}_1 + \boldsymbol{M}_4\boldsymbol{V}_2) + (\boldsymbol{\xi}^{u,E} \circ u'(\boldsymbol{c}))^\top\boldsymbol{\Pi}\boldsymbol{V}_2$$
$$\boldsymbol{L}_1 = \left(\tilde{\boldsymbol{a}}^\top(\boldsymbol{M}_3 + \boldsymbol{M}_4\boldsymbol{M}_5) + (\boldsymbol{\xi}^{u,E} \circ u'(\boldsymbol{c}))^\top\boldsymbol{\Pi}\boldsymbol{M}_5\right)/C_1$$

Then:

$$\mu = -\boldsymbol{L}_1\bar{\omega}$$

We deduce that from (198) and (199):

$$\begin{aligned}
\bar{\boldsymbol{\lambda}}_c &= (\boldsymbol{M}_5 - \boldsymbol{V}_2\boldsymbol{L}_1)\bar{\boldsymbol{\omega}}, \\
\bar{\boldsymbol{\psi}} &= \boldsymbol{M}_3\bar{\boldsymbol{\omega}} + \boldsymbol{M}_4\bar{\boldsymbol{\lambda}}_c + \mu\boldsymbol{V}_1 \\
&= (\boldsymbol{M}_3 + \boldsymbol{M}_4(\boldsymbol{M}_5 - \boldsymbol{V}_2\boldsymbol{L}_1) - \boldsymbol{V}_1\boldsymbol{L}_1)\bar{\boldsymbol{\omega}} \\
&= \boldsymbol{M}_6\bar{\boldsymbol{\omega}},
\end{aligned}$$

with:

$$\boldsymbol{M}_6 = \boldsymbol{M}_3 + \boldsymbol{M}_4(\boldsymbol{M}_5 - \boldsymbol{V}_2\boldsymbol{L}_1) - \boldsymbol{V}_1\boldsymbol{L}_1.$$

and from (196):

$$\bar{\boldsymbol{\lambda}}_l = \hat{\boldsymbol{M}}_6\bar{\boldsymbol{\omega}}$$
$$\text{where: } \hat{\boldsymbol{M}}_6 = \boldsymbol{M}_0 + \boldsymbol{M}_1\boldsymbol{M}_6 - \boldsymbol{V}_0\boldsymbol{L}_1,$$

Constructing the two vectors $\boldsymbol{L}_2$ and $\boldsymbol{L}_3$ The constraint of equation (195) is:

$$(\ln(\boldsymbol{y} \circ \boldsymbol{l}) \circ \boldsymbol{\xi}^y \circ (\boldsymbol{y} \circ \boldsymbol{l})^\tau)^\top\bar{\boldsymbol{\psi}} = -\left((\boldsymbol{1} + \tau\ln(\boldsymbol{y} \circ \boldsymbol{l})) \circ \boldsymbol{\xi}^y \circ (\boldsymbol{y} \circ \boldsymbol{l})^\tau \circ \tilde{\boldsymbol{\xi}}^{u,1} \circ u'(\boldsymbol{c})\right)^\top\bar{\boldsymbol{\lambda}}_l,$$

or

$$(\ln(\boldsymbol{y} \circ \boldsymbol{l}) \circ \boldsymbol{\xi}^y \circ (\boldsymbol{y} \circ \boldsymbol{l})^\tau)^\top\boldsymbol{M}_6\bar{\boldsymbol{\omega}} = -\left((\boldsymbol{1} + \tau\ln(\boldsymbol{y} \circ \boldsymbol{l})) \circ \boldsymbol{\xi}^y \circ (\boldsymbol{y} \circ \boldsymbol{l})^\tau \circ \tilde{\boldsymbol{\xi}}^{u,1} \circ u'(\boldsymbol{c})\right)^\top\hat{\boldsymbol{M}}_6\bar{\boldsymbol{\omega}},$$

or equivalently:

$$\boldsymbol{L}_2\bar{\boldsymbol{\omega}} = 0$$
$$\boldsymbol{L}_2 = (\ln(\boldsymbol{y} \circ \boldsymbol{l}) \circ \boldsymbol{\xi}^y \circ (\boldsymbol{y} \circ \boldsymbol{l})^\tau)^\top\boldsymbol{M}_6 + \left((\boldsymbol{1} + \tau\ln(\boldsymbol{y} \circ \boldsymbol{l})) \circ \boldsymbol{\xi}^y \circ (\boldsymbol{y} \circ \boldsymbol{l})^\tau \circ \tilde{\boldsymbol{\xi}}^{u,1} \circ u'(\boldsymbol{c})\right)^\top\hat{\boldsymbol{M}}_6$$

The constraint (194) stating

$$\left(\boldsymbol{\xi}^y \circ (\boldsymbol{y} \circ \boldsymbol{l})^\tau\right)^\top \bar{\boldsymbol{\psi}} = -\tau \left(\boldsymbol{\xi}^y \circ (\boldsymbol{y} \circ \boldsymbol{l})^\tau \circ \tilde{\boldsymbol{\xi}}^{u,1} \circ u'(\boldsymbol{c})\right)^\top \bar{\boldsymbol{\lambda}}_l,$$

becomes:

$$\begin{aligned} \boldsymbol{L}_3 \bar{\boldsymbol{\omega}} &= 0 \\ \text{with: } \boldsymbol{L}_3 &= \left(\boldsymbol{\xi}^y \circ (\boldsymbol{y} \circ \boldsymbol{l})^\tau\right)^\top \boldsymbol{M}_6 + \tau \left(\boldsymbol{\xi}^y \circ (\boldsymbol{y} \circ \boldsymbol{l})^\tau \circ \tilde{\boldsymbol{\xi}}^{u,1} \circ u'(\boldsymbol{c})\right)^\top \hat{\boldsymbol{M}}_6. \end{aligned}$$

D.4 Estimating weights

We assume that for each history with the same current productivity has the same weight. As a consequence, there are K different Pareto weights, ω^s, for all histories.

Define $\boldsymbol{M}_7$ as the $N_{tot} \times K$ matrix that allocates the productivity weights to the N_{tot} histories. Elements of $\boldsymbol{M}_7$ are only $0, 1$ and the line $i = 1...N_{tot}$ has a one in column $j = 1...K$ if and only if the current productivity of history i is y_j.

Considering the size of the history, the vectors of weights by history is:

$$\bar{\boldsymbol{\omega}} = \boldsymbol{D}_{\boldsymbol{S}} \boldsymbol{M}_7 \boldsymbol{\omega}^s$$

Finally, the estimated Pareto weights are those which are close to 1 (the utilitarian Social Welfare Function), such that the first-order conditions of the planner are fulfilled. We thus solve:

$$\begin{aligned} &\min_{\omega} \left\| \boldsymbol{\omega}^s - \mathbf{1}_K \right\|^2, \\ &\text{s.t. } \boldsymbol{L}_2 \boldsymbol{D}_{\boldsymbol{S}} \boldsymbol{M}_7 \boldsymbol{\omega}^s = 0 \\ &\phantom{\text{s.t. }} \boldsymbol{L}_3 \boldsymbol{D}_{\boldsymbol{S}} \boldsymbol{M}_7 \boldsymbol{\omega}^s = 0 \end{aligned}$$

We now provide the simple solution to this linear-quadratic problem. We solve:

$$\begin{aligned} \max_{\boldsymbol{\omega}^s} \; (\boldsymbol{\omega}^s - \mathbf{1}_K)^\top (\boldsymbol{\omega}^s - \mathbf{1}_K) &- 2\mu_2 (\boldsymbol{L}_2 \boldsymbol{D}_{\boldsymbol{S}} \boldsymbol{M}_7 \boldsymbol{\omega}^s) \\ &- 2\mu_3 (\boldsymbol{L}_3 \boldsymbol{D}_{\boldsymbol{S}} \boldsymbol{M}_7 \boldsymbol{\omega}^s) \end{aligned}$$

The FOCs of the problem are:

$$\begin{aligned}
\boldsymbol{\omega}^s - \mathbf{1}_K &= \sum_{k=2}^{3} \mu_k (\boldsymbol{L}_k \boldsymbol{D}_{\boldsymbol{S}} \boldsymbol{M}_7)^\top \\
-\boldsymbol{L}_2 \boldsymbol{D}_{\boldsymbol{S}} \boldsymbol{M}_7 \mathbf{1}_K &= \sum_{k=2}^{3} \mu_k \boldsymbol{L}_2 \boldsymbol{D}_{\boldsymbol{S}} \boldsymbol{M}_7 (\boldsymbol{L}_k \boldsymbol{D}_{\boldsymbol{S}} \boldsymbol{M}_7)^\top \\
-\boldsymbol{L}_3 \boldsymbol{D}_{\boldsymbol{S}} \boldsymbol{M}_7 \mathbf{1}_K &= \sum_{k=2}^{3} \mu_k \boldsymbol{L}_3 \boldsymbol{D}_{\boldsymbol{S}} \boldsymbol{M}_7 (\boldsymbol{L}_k \boldsymbol{D}_{\boldsymbol{S}} \boldsymbol{M}_7)^\top \\
-\boldsymbol{L}_4 \boldsymbol{D}_{\boldsymbol{S}} \boldsymbol{M}_7 \mathbf{1}_K &= \sum_{k=2}^{3} \mu_k \boldsymbol{L}_4 \boldsymbol{D}_{\boldsymbol{S}} \boldsymbol{M}_7 (\boldsymbol{L}_k \boldsymbol{D}_{\boldsymbol{S}} \boldsymbol{M}_7)^\top
\end{aligned}$$

Then:

$$\begin{aligned}
\boldsymbol{M}_8 &= \begin{bmatrix} \boldsymbol{L}_2 \boldsymbol{D}_{\boldsymbol{S}} \boldsymbol{M}_7 \\ \boldsymbol{L}_3 \boldsymbol{D}_{\boldsymbol{S}} \boldsymbol{M}_7 \end{bmatrix} \begin{bmatrix} \boldsymbol{L}_2 \boldsymbol{D}_{\boldsymbol{S}} \boldsymbol{M}_7 \\ \boldsymbol{L}_3 \boldsymbol{D}_{\boldsymbol{S}} \boldsymbol{M}_7 \end{bmatrix}^\top \\
\boldsymbol{V}_8 &= - \begin{bmatrix} \boldsymbol{L}_2 \boldsymbol{D}_{\boldsymbol{S}} \boldsymbol{M}_7 \mathbf{1}_K \\ \boldsymbol{L}_3 \boldsymbol{D}_{\boldsymbol{S}} \boldsymbol{M}_7 \mathbf{1}_K \end{bmatrix}
\end{aligned}$$

and:

$$\begin{bmatrix} \mu_2 \\ \mu_3 \end{bmatrix} = \boldsymbol{M}_8^{-1} \boldsymbol{V}_8.$$

and, finally

$$\begin{aligned}
\boldsymbol{\omega}^s &= \mathbf{1}_K + \sum_{k=2}^{3} \mu_k (\boldsymbol{L}_k \boldsymbol{D}_{\boldsymbol{S}} \boldsymbol{M}_7)^\top \\
&= \mathbf{1}_K + \begin{bmatrix} \mu_2 \\ \mu_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{L}_2 \boldsymbol{D}_{\boldsymbol{S}} \boldsymbol{M}_7 \\ \boldsymbol{L}_3 \boldsymbol{D}_{\boldsymbol{S}} \boldsymbol{M}_7 \end{bmatrix}^\top \\
&= \mathbf{1}_K + \boldsymbol{M}_8^{-1} \boldsymbol{V}_8 \begin{bmatrix} \boldsymbol{L}_2 \boldsymbol{D}_{\boldsymbol{S}} \boldsymbol{M}_7 \\ \boldsymbol{L}_3 \boldsymbol{D}_{\boldsymbol{S}} \boldsymbol{M}_7 \end{bmatrix}^\top
\end{aligned}$$

E Dynamic system to be simulated

The dynamics of the allocation for given instruments of the planner are:

$$
\begin{aligned}
&G_t + T_t + (1+r_t)B_{t-1} + r_t K_{t-1} + w_t \sum_{k=1}^{N_{tot}} S_k \xi_k^y (l_{t,k} y_k)^{1-\tau_t} = F(K_{t-1}, L_t) + B_t, \\
&\text{for all } k = 1 \ldots N_{tot}\text{: } , c_{t,k} + a_{t,k} = w_t \xi_k^y (l_{t,k} y_k)^{1-\tau_t} + (1+r_t)\tilde{a}_{t,k} + T_t, \\
&\qquad a_{t,k} \geq -\bar{a},\ \nu_{t,k}(a_{t,k} + \bar{a}) = 0,\ \nu_{t,k} \geq 0, \\
&\qquad \xi_k^{u,E} u'(c_{t,k}) = \beta \mathbb{E}_t \Big[\sum_{\tilde{k}=1}^{N_{tot}} \Pi_{t,k\tilde{k}} \xi_{\tilde{k}}^{u,E} u'(c_{t+1,\tilde{k}})(1+r_{t+1}) \Big] + \nu_{t,k}, \\
&\qquad \xi_{t,k}^{v,1} v'(l_{t,k}) \equiv \tau_t w_t \xi_k^y (l_{t,k} y_k)^{1-\tau_t} \xi_k^{u,1} (u'(c_{t,k})/l_{t,k}), \\
&\qquad K_t + B_t = \sum_{k=1}^{N_{tot}} S_k a_{t,k},\ L_t = \sum_{k=1}^{N_{tot}} S_k y_k l_{t,k}.
\end{aligned}
$$

The dynamic equations characterizing the solution of the planner for the truncated model are, for $k = 1 \ldots N_{tot}$:

$$
\begin{aligned}
\left(S_k \hat{\psi}_{t,k}\right) \;=\;& \left(S_k \omega_k\right) \xi_k^{u,0} u'(c_{t,k}) - S_k \mu_t \\
& - \left(\left(S_k \lambda_{c,t,k}\right) \xi_k^{u,E} - (1+r_t) \left(S_k \tilde{\lambda}_{c,t,k}\right) \xi_k^{u,E} - \left(S_k \lambda_{l,t,k}\right) \xi_k^y \tau_t w_t (y_k)^{\tau_t} l_{t,k}^{\tau_t - 1} \xi_k^{u,1} \right) u''(c_{t,k}).
\end{aligned}
$$

$$
\left(S_k \hat{\psi}_{t,k}\right)/S_k = \beta \mathbb{E}_t \Big[(1+r_{t+1}) \sum_{\tilde{k}=1}^{N_{tot}} \Pi_{t,k\tilde{k}} \left(S_{\tilde{k}} \hat{\psi}_{t+1,\tilde{k}}\right)/S_{\tilde{k}} \Big] \text{ if } \nu_k = 0 \text{ and } \lambda_{t,k} = 0 \text{ otherwise,} \tag{200}
$$

$$\frac{1}{\tau_t w_t \xi_k^{\tau}(y_k)^{\tau_t} l_{t,k}^{\tau_t-1}}\left(\left(S_k\omega_k\right)\xi_k^{v,0}v'(l_{t,k})+\left(S_k\lambda_{l,t,k}\right)\xi_k^{v,1}v''(l_{t,k})\right)=\left(S_k\hat{\psi}_{t,k}\right) \tag{201}$$

$$+\left(S_k\lambda_{l,t,k}\right)(\tau_t-1)\xi_k^{u,1}(u'(c_{t,k})/l_{t,k}) \tag{202}$$

$$+S_k\mu_t(1-\alpha)\frac{Y_t}{\tau_t w_t \xi_k^{\tau}(y_k)^{\tau_t-1} l_{t,k}^{\tau_t-1} L_t} \tag{203}$$

$$\sum_{k=1}^{N_{tot}}\left(\left(S_k\hat{\psi}_{t,k}\right)\tilde{a}_{t,k}+\left(S_k\tilde{\lambda}_{c,t,k}\right)\xi_k^{u,E}u'(c_{t,k})\right)=0 \tag{204}$$

$$\sum_{k=1}^{N_{tot}}\xi_k^{\tau}(l_{t,k}y_k)^{\tau_t}\left(\left(S_k\hat{\psi}_{t,k}\right)+\left(S_k\lambda_{l,t,k}\right)\tau_t\xi_k^{u,1}(u'(c_{t,k})/l_{t,k})\right)=0 \tag{205}$$

$$\mu_t=\beta\mathbb{E}\left[\mu_{t+1}\left(1+\alpha\frac{Y_{t+1}}{K_t}-\delta\right)\right] \tag{206}$$

$$0=\sum_{k=1}^{N_{tot}}\left(\left(S_k\hat{\psi}_{t,k}\right)+\left(S_k\lambda_{l,t,k}\right)\tau_t\xi_k^{u,1}(u'(c_{t,k})/l_{t,k})\right)\ln\left(l_{t,k}y_k\right)\xi_k^{\tau}(l_{t,k}y_k)^{\tau_t} \tag{207}$$

$$+\sum_{k=1}^{N_{tot}}\left(S_k\lambda_{l,t,k}\right)\xi_k^{\tau}(l_{t,k}y_k)^{\tau_t}\xi_k^{u,1}(u'(c_{t,k})/l_{t,k}) \tag{208}$$

The set of all equations can be simulated on Dynare.

F US spending shocks

We use the data collected by Ramey and Zubairy (2018) to analyze public spending shocks. As Ramey and Zubairy (2018), we normalize macro variables by real potential GDP, based on 6th degree polynomial fit from 1889:1–2015:4, omitting Great Depression and WWII. We use then remove a linear trend from public spending data. We estimate the process for public spending considering the period from the peak of spending to the end of the event. The public debt measure is Public debt held by the public. To compute the discounted value of public spending we use the real market 3 months interest rate, using 3-month treasury bill rate on secondary market and removing realized inflation. As Ramey and Zubairy (2018) for the beginning of the sample, we use NY discount rate for early years. For the year 1914, we use the rate of 1915 due to missing data.